Math 671 - Midterm

Assigned March 9, 2002-Due March 21, 2002

Q1. A function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is called a *stuttering* function, if f is monotone nondecreasing and increases by increments of at most 1 at a time. More formally, if $f(0) = 0$ and $f(m+1)$ satisfies $f(m) \leq f(m+1) \leq f(m) + 1$ for all $m \geq 0$. Given a run r, we say that r' is a *stuttered version* of r if, for some stuttering function f, we have $r'(m) = r(f(m))$ for all $m \geq 0$. Prove that a.m.p. systems are closed under stuttering, that is, if r is a run of an a.m.p. system $\mathcal{R}$ and r' is a stuttered version of r, then r' is a run of $\mathcal{R}$. Note that from this it follows that any run in which there are arbitrarily long "silent intervals" between the events of r will also be in $\mathcal{R}$

Q2. Suppose r is a run in an a.m.p. system $\mathcal{R}$, and e is an event, and that k is a positive integer. Let r' be defined as follows: let i be an agent, if there is no event e' and time m_i where $e \xrightarrow{r} e'$ and e' is in $r_i(m_i)$, then let $r'_i(m) = r_i(m)$ for every m. Otherwise, let m_i be minimal such that there is an event e' where $e \xrightarrow{r} e'$ and e' is in $r_i(m_i)$. Then define

$$r'_i(m) = \begin{cases} r_i(m) & \text{if } m < m_i \\ r_i(m_i - 1) & \text{if } m_i \leq m < m_i + k \\ r_i(m - k) & \text{if } m_i + k \leq m. \end{cases}$$

Show that r' is a run in $\mathcal{R}$.

Q3. Prove the following proposition:

Proposition 0.1 *Let G be the group of all processes, let $\mathcal{R}$ be an a.m.p. system, and assume that the interpretation of $Prec(e, e')$ in $\mathcal{I} = (\mathcal{R}, \pi)$ is standard. Then $(\mathcal{I}, r, m) \models D_G(Prec(e, e'))$ iff e and e' have both occurred by round m of r and $e \xrightarrow{r} e'$.*

(Hint: if part is straightforward. For the only if part, if it is not the case that $e \xrightarrow{r} e'$ use Q2 above to find a run $r' \in \mathcal{R}$ and a time m' such that $(r, m) \sim_i (r', m')$ for all processes i and $(\mathcal{I}, r', m') \models \neg Prec(e, e')$.)

Q4. (Selfish and altruistic social behaviour) Two people enter a bus. Two adjacent cramped seats are free. Each person must decide whether to sit or stand. Sitting alone is more comfortable than sitting next to the other person, which is more comfortable than standing.

a. Suppose that each person cares only about her own comfort. Model the situation as a strategic game. Is this game the *Prisoner's Dilemma*? Find its Nash equilibrium (equilibria?).

b. Suppose that each person is altruistic, ranking the outcomes according to the *other* person's comfort, and , out of politeness, prefers to stand than to sit if the other person stands. Model the situation as a strategic game. Is this game the *Prisoner's Dilemma*? Find its Nash equilibrium (equilibria?).

c. Compare the people's comfort in the equilibria of the two games.

Q5. (Contributing to a public good) Each of n people chooses whether or not to contribute a fixed amount toward the provision of a public good. The good is provided if and only if at least k people contribute, where $2 \leq k \leq n$; if it is not provided, contributions are not refunded . Each person ranks outcomes from best to worst as follows: (i) any outcome in which the good is provided and she does not contribute, (ii) any outcome in which the good is provided and she contributes, (iii) any outcome in which the good is not provided and she does not contribute, (iv) any outcome in which the good is not provided and she contributes. Formulate this situation as a strategic game and find its Nash equilibria. Is there a Nash equilibrium in which more than k people contribute? One in which k people contribute? One in which fewer than k people contribute?

Q6. (An auction) An object is to be assigned to a player in the set $\{1, 2, \ldots, n\}$ in exchange for a payment. Player i's valuation of the object is v_i and $v_1 > v_2 > \cdots > v_n > 0$. The mechanism used to assign the object is a sealed-bid auction: the players simultaneously submit bids (nonnegative numbers), and the object is given to the player with the lowest index among those who submit the highest bid, in exchange for a payment. In a *first price* auction the payment that the winner makes is the price that he bids. Formulate a first price auction as a strategic game and find its Nash equilibria. In particular, show that in all equilibria player 1 obtains the object.

Q7. Recall the example of a Bayesian game that we looked at in class (BoS game). Complete the construction of G^* that we started in class and find all Nash equilibria. Now suppose player 1 and 2 have prior beliefs as $p_i(b, s) = p_i(s, b) = 1/8$ and $p_i(s, s) = 1/2$ and $p_i(b, b) = 1/4$. Construct G^* and find all Nash equilibria.

Q8. Show that more information may hurt a player by constructing a two-player Bayesian game with the following features. Player 1 is fully informed while player 2 is not; the game has a unique Nash equilibrium, in which player 2's payoff is higher than his payoff in the unique equilibrium of any of the related games in which he knows player 1's type. (Assume $u_1, u_2 : A \times \Omega \longrightarrow \mathbb{R}$.)