On Köthe sequence spaces and linear logic

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Abstract

We present a category of locally convex topological vector spaces which is a model of propositional classical linear logic, based on the standard concept of Köthe sequence spaces. In this setting, the spaces interpreting the exponential have a quite simple structure of commutative Hopf algebra. The co-Kleisli category of this linear category is a cartesian closed category of entire mappings. This work provides a simple setting where typed λ-calculus and differential calculus can be combined; we give a few examples of computations.

1 Introduction

It has been clear from the very beginning that linear logic has something to do with linear algebra, and thus that, at some point, some real or complex coefficients should naturally appear. Nevertheless most concrete models discovered so far are essentially *discrete* (coherence spaces, hypercoherences, games...). Interpreting formulae of linear logic as vector spaces is not easy because to formulae containing exponentials one must associate infinite dimensional spaces and for that reason, *topological vector spaces* must be considered. Several models based on topological vector spaces have already been presented, by Girard [Gir99] and Blute [Blu96], in a setting considered first by Barr in [Bar79, Bar90]. Other semantics of typed lambda-calculi with real coefficients have been considered, among which we would like to mention in particular Danos and Harmer's probabilistic games [DH00], whose non-deterministic character seems to be very much in the spirit of the model presented here.

We propose a quite concrete approach based on a standard notion in the theory of locally convex topological vector spaces (lcs for short), the notion of *Köthe sequence spaces* (see [Köt66]). These spaces constitute a well-behaved class of Hausdorff and complete lcs which are not Banach spaces in general and, with this respect, our approach will radically differ from Girard's model of coherence Banach spaces. Another difference is the absence of any notion of coherence in our model, whereas in Girard's setting, "coherent" vectors are those whose norm is less than 1. Our semantics will also be quite different from Blute's model as the topology we shall consider on the underlying field (the field of real numbers, or the field of complex numbers) will be the standard topology, not the discrete one. In [Bar79], chapter 4, example 1, topological vector spaces over the field of real or complex numbers with the usual topology are also considered, and the construction presented in that book (and simplified later with the help of Chu spaces) yields a $\star$-autonomous category (a model of multiplicative-additive linear logic) which, using the constructions of [Bar90], should lead to a model of full propositional linear logic based on topological vector spaces. These constructions however are fairly abstract, and we do not know whether a more concrete description of the resulting model is possible.

The central notion of the model under consideration is the concept of an *absolutely converging sum*. For understanding the relevance of this concept to denotational semantics, one must consider settings where the exponentials of linear logic are interpreted in a *non-uniform way*, as in the models induced by the symmetric product phase spaces of [BE99b]. The usual coherence space semantics indeed is *uniform*: this means that the elements of the web of the coherence space $!E$ are the finite *cliques* (or multi-cliques) of the coherence space E. For that reason, this semantics has the essential property that the intersection of a clique and an anti-clique contains at most one element. Intuitively, in a language inspired by game models, the interaction between a strategy and a counter-strategy yields at most one result: this is a property of *determinism*. As explained in [BE99b], in a non-uniform version of coherence spaces, the intersection of a clique and an anti-clique can have more than one element. This is due to the fact that, in this setting, two distinct points of a web can be *neutral* (that is, neither coherent nor incoherent) without being equal.

The basic "algebraic" intuition behind coherence spaces is that cliques of E should be considered as "vectors" and that anti-cliques of E (that is, cliques of $E^\perp$) should be considered as "linear forms" on E; indeed, $E^\perp$ is isomorphic to $E \multimap \perp$, the space of linear maps from E to $\perp$ (the one point coherence space, which plays the rôle of the "field" considered as a space). If we consider a subset of $|E|$ as a $|E|$-indexed family of elements of $\{0, 1\}$ (the "field" considered as the set of scalars[1]), then given a clique x of E and an anti-clique x' of E, their intersection has 0 or 1 element, and its cardinality is trivially given by

$$\#(x \cap x') = \sum_{a \in |E|} x_a x'_a \,,$$

this sum converging because at most one of its terms is different from 0(!). In a non-uniform setting, there is no reason why this sum should converge: in the non-uniform coherence spaces mentioned above, we just know that if $x_a x'_a = 1$ and $x_b x'_b = 1$, then a and b are neutral, but they can be different. It is here that Köthe spaces come in: let us replace the "field" $\{0, 1\}$ by one of the two fields $\mathbf{R}$ or $\mathbf{C}$ (denoted by $\mathbf{K}$), and let us take as a basic requirement for building our spaces the convergence of the sum above. Another solution consists in staying in the discrete setting (still considering subsets of the webs) and saying that two subsets are orthogonal if their intersection is finite; this leads to the notion of *finiteness spaces*, see [Ehr00].

More precisely, an at most countable set I being given (the web), we replace the notion of "subset of I" by the notion of "element of $\mathbf{K}^I$". Then, given $x, x' \in \mathbf{K}^I$ we replace the notion of "cardinality of the intersection of x and x'" by the notion of "value of the series $\sum_{i \in I} x_i x'_i$". This latter value of course is not always well defined; since there is *a priori* no order relation on I, the only reasonable condition to ask on x and x' for this series to converge is to require its absolute convergence[2]. So, following Girard's terminology (which, in the present context, could be misleading, and therefore will be avoided later on), let us say that $x, x' \in \mathbf{K}^I$ are *orthogonal* if the series $\sum_{i \in I} |x_i x'_i|$ converges, that is, if $xx' \in \ell^1(I)$. Then the notion of "coherence space of web I" will be replaced by the notion of subset E of $\mathbf{K}^I$ such that $E = E^{\perp\perp}$ for this orthogonality, where $E^\perp$ denotes the set of all the elements of $\mathbf{K}^I$ which are orthogonal to all the elements of E[3].

[1]It is definitely not a field in the usual sense, we just take the word "field" for developping an analogy.

[2]A classical result due to Riemann asserts that if a family $(x_i)_{i \in \mathbf{N}}$ is such that $\sum_{i \in \mathbf{N}} x_{\sigma(i)}$ converges for each permutation σ of the natural numbers, then the series $\sum_{i \in \mathbf{N}} x_i$ converges absolutely.

[3]Interestingly enough, coherence spaces can also be defined in this way: say that $x, x' \subseteq I$ are orthogonal if $\#(x \cap x') \leq 1$, then a coherence space of web I can equivalently be defined as a subset of $\mathcal{P}(I)$ which is equal to its bi-orthogonal.

It turns out that such a subset E of $\mathbf{K}^I$ is always a vector space (with operations defined in the obvious componentwise manner), and even a locally convex topological vector space which is Hausdorff and complete (its topology is defined by a family of semi-norms induced by the elements of $E^\perp$): these spaces are *Köthe sequence spaces*[4] (which are classically defined as the orthogonals of certain subsets of $\mathbf{K}^I$ called *Köthe sets*), a notion due to Köthe and Toeplitz; see [Jar81], which will be our basic reference for locally convex spaces. Another classical reference book on this topic is [Sch73], see chapter 4 for a few informations on Köthe sequence spaces. More detailed informations on this topic can also be found in [Pie72] and of course in [Köt66] (in German). The topological vector spaces so defined are not always (more precisely, they are almost never) Banach spaces, they can even not be metrizable, but they behave remarkably well with respect to the operations needed for interpreting linear logic. The sum operation in these spaces corresponds intuitively to the superimposition of various possible results of a (non-deterministic) computation: this use of the sum operation has already been considered by Arbib and Manes in [AM86], see also [Hag01] where Haghverdi uses sums (at the "geometry of interaction" level) for building fully complete models of multiplicative linear logic. Possible connections between our model and Haghverdi's constructions have still to be explored.

For us, a *Köthe space* is a pair $X = (|X|, E_X)$ where $|X|$ is an at most denumerable set and E_X is a subset of $\mathbf{K}^{|X|}$ such that ${E_X}^{\perp\perp} = E_X$. So the space is given together with $|X|$ (its *web*; the word has not to be taken in the sense it has in the theory of locally convex spaces), that is, with an explicit topological Schauder basis (see [Sch73] chapter 3 and [Jar81] chapter 14) which is actually an *absolute* topological basis (see [Jar81] chapter 14), and we shall heavily use this basis in our constructions. A morphism from X to Y is a linear continuous function from E_X to E_Y, and we show that these functions are in bijective correspondence with the elements of $E_{X\multimap Y}$ where $X \multimap Y$ is a Köthe space whose web is $|X| \times |Y|$; the element of $E_{X\multimap Y}$ corresponding to a linear continuous function is its (infinite-dimensional) *matrix*. We also define a tensor product of Köthe spaces and show that the category of Köthe spaces and continuous linear maps is a $\star$-autonomous category with (at most countable) sums and products, and we obtain in that way a model of multiplicative additive linear logic. One of the most striking features of this semantics is *non-determinism*. Typically, a boolean value will not simply be $\mathbf{t}$ (true) of $\mathbf{f}$ (false), but an arbitrary linear combination $\lambda\mathbf{t} + \mu\mathbf{f}$ of these two values, with coefficients $\lambda, \mu \in \mathbf{K}$.

Next we introduce the exponential $!X$ of a Köthe space X, which turns out to have a quite rich algebraic structure. As in [Gir99], the vector space $E_{!X\multimap Y}$ will be isomorphic to a space of analytic mappings taking their values in E_Y, but here, these mappings will be defined on the whole space E_X (and will actually be *entire* functions). In [Gir99] indeed, the analytic functions where defined only on the open unit ball, that is, on a set of "coherent" vectors: in that sense, the semantics of coherence Banach spaces is uniform whereas ours is not. Take $Y = \perp$ (that is, $E_Y = \mathbf{K}$) for simplicity. The web of $!X$ will be, as it is standard in the semantics of linear logic, the set of all finite multi-sets on $|X|$ (a multi-set on a set I is a map from I to $\mathbf{N}$, it is finite if it vanishes almost everywhere). Given $x \in E_X$ and $\mu \in |!X|$, we define x^μ as the finite product $\prod_{a\in|X|} x_a^{\mu(a)}$: this is just the standard notion of a multi-exponent used in the theory of polynomials (or entire functions) of several variables (the rôle of the variables being played by the elements of the web), and we define $E_{!X}$ simply as the biorthogonal of the set of all families $(x^\mu)_{\mu\in|!X|}$ when $x \in E_X$. Equivalently, $E_{(!X)^\perp}$ is the set of all families $(u_\mu)_{\mu\in|!X|}$ of scalars such that the series $\sum_{\mu\in|!X|} u_\mu x^\mu$ converges absolutely, for all $x \in E_X$, that is, the space of all power series which converge on the whole space

[4] It is worth observing that the basic "pre-$\star$-autonomous situations" considered in [Bar79], chapter 4, example 1, are classes of Köthe sequence spaces, suggesting possible connections between the model we present here and the above mentioned model due to Barr.

E_X. To give a concrete example, when $X = 1$ (again, $|X|$ is a singleton, so $E_X = \mathbf{K}$), $|!X| = \mathbf{N}$ and $E_{(!X)^\perp}$ is the space of all power series of infinite convergence radius on $\mathbf{K}$. Coming back to the general situation, an entire map from X to Y (Köthe spaces) is by definition a function from E_X to E_Y which is definable by such a series (which is then necessarily unique). This interpretation of intuitionistic proofs as entire maps is of course completely in the spirit of Girard's quantitative semantics (see [Gir88, BE99a], and also [Has97] where a method is developed for computing the coefficients of the "power series" associated to λ-terms in Girard's quantitative semantics).

The space $!X$ has a co-algebraic structure, as it is standard in the semantics of linear logic (this structure is used for interpreting contraction and weakening), but also an algebraic structure, and is actually a quite simple commutative Hopf algebra. The algebraic structure is used e.g. for computing the derivatives of entire maps. The operation $X \mapsto !X$ is functorial, and defines as usual a comonad on the category of Köthe spaces and continuous linear maps satisfying the properties required for interpreting the ! modality of linear logic. The co-Kleisli category of this co-monad is therefore cartesian closed, and is isomorphic to the category of Köthe spaces and entire maps. This may seem surprising *a priori* as one is used to the idea that continuity on Hausdorff spaces, and so *a fortiori* analyticity, is incompatible with cartesian closeness[5]. The point is that our entire maps *are not continuous* with respect to the native topology of the Köthe spaces on which they are defined; continuity with respect to this topology is relevant only for linear maps. Typically, the bilinear evaluation function $E_X \times E_{X^\perp}$ which maps (x, x') to $\langle x, x' \rangle = \sum_{a \in |X|} x_a x'_a$ is not continuous with respect to the product topology, as soon as the set $|X|$ is infinite. Although this phenomenon may seem weird, it is completely standard in the theory of locally convex spaces. For instance, the extension of differentiability to infinite dimensional vector spaces developed by Frölicher and Kriegl in [FK88] is apparently based on the idea that smoothness is not a topological notion, but a "bornological" notion. This viewpoint is exploited in the book by Kriegl and Michor [KM97].

At the end of the paper, we sketch a theory of *intrinsic Köthe spaces* for advocating the fact that, although we used intensively the webs in our space constructions, the spaces obtained do not really depend on them. We define an intrinsic Köthe space as a topological vector space E which is linearly homeomorphic to some Köthe space: this is a *property of the topology of* E. Using the functoriality of the operations defined on Köthe spaces, we show how each operation on Köthe spaces has a corresponding operation on intrinsic Köthe spaces. For instance, given E and F intrinsic Köthe spaces, we endow $\mathcal{L}(E, F)$ (the vector space of linear continuous maps from E to F) with a topology such that this space becomes an intrinsic Köthe space. For defining this topology, we use linear homeomorphisms of E and F to Köthe spaces, but the resulting topology *does not depend on this choice.* Of course, it would be much more satisfactory to directly define this topology in terms of the topologies of E and F, but we do not know how to do that in general yet (we give a negative result which closes the most natural track). Note that an isomorphism between E and some Köthe space E_X can also be seen as a choice in E of an absolute basis satisfying some additional requirement (corresponding to the fact that $E_X = {E_X}^{\perp\perp}$); we choose such bases in E and F for defining the topology of $\mathcal{L}(E, F)$, although this topology does not depend on this choice.

Of course, the present work does not pretend to bring any new idea to the rich theory of functional analysis and topological vector spaces. Our approach is intensionally quite concrete (bases are used everywhere), we do not seek for generality; for instance in our spaces of analytic mappings, we only consider mappings defined on the whole space by a power series having an "infinite radius of convergence" (entire functions), although Köthe spaces are probably more expressive than that. We just want to illustrate the fact that rather simple topological vector spaces can quite easily be

[5]See [Mac71], chapter 7, section 8 and also the introduction of [Gir99].

used for modeling standard linear logic and typed λ-calculus[6]. The paper remains at an elementary level, and no deep knowledge of the theory of topological vector spaces is required; this is due to the omni-presence of webs (canonical bases). Many important questions remain unanswered among which we can mention: characterization of the topology in linear and entire function spaces, characterization of the tensor product and comparison with the standard tensor products on lcs. They are postponed to future work.

Contents

2 Köthe spaces

We denote by $\mathbf{K}$ the field of real or complex numbers endowed with its usual topology. Let I be an at most denumerable set. A *sequence space* is a sub-vector space of K^I which contains all sequences u such that $u_i = 0$ for almost all i.

[6]In [Gir99], linear logic is modified so as to make its interpretation in the category of coherence Banach spaces possible, by the introduction of *weighted sequents*. This is due to the fact that Girard's analytic functions are defined on the *open* unit ball whereas proofs are interpreted in the *closed* unit ball; weights are used for enabling the application of an analytic function to the interpretation of a proof. This discrepancy seems essentially due to the fact that his analytic maps are required to be continuous with respect to the topology of the Banach spaces on which they are defined.

For the sake of self-containedness, we recall first some simple informations about infinite sums. A family $u = (u_i)_{i\in I}$ of non-negative real numbers is *summable* if the set $\{\sum_{j\in J} u_j \mid J \in \mathcal{F}\} \subseteq \mathbf{R}^+$ (where $\mathcal{F}$ is the set of all finite subsets of I) is bounded, and then $\sum_{i\in I} u_i$ denotes the least upper bound of this set. A family $u = (u_i)_{i\in I}$ of elements of $\mathbf{K}$ is *absolutely summable* if the family $(|u_i|)_{i\in I}$ of non-negative real numbers is summable. In that case, the net $(\sum_{i\in J} u_j)_{J\in\mathcal{F}}$ converges to an element of $\mathbf{K}$ denoted by $\sum_{i\in I} u_i$. This sum can also be defined as the sum of the absolutely convergent series $\sum_{n=0}^{\infty} u_{i_n}$ where $(i_n)_{n\in\mathbf{N}}$ is an arbitrary enumeration without repetitions of I (assuming I to be infinite). For absolutely summable families, summing is an associative and commutative operation.

Duality. If $u, v \in \mathbf{K}^I$, we shall say that u and v are *in duality* if the sum $\sum_{i\in I} |u_i v_i|$ converges, that is, if $uv \in \ell^1(I)$. In that case, we shall denote by $\langle u, v\rangle$ the sum $\sum_{i\in I} u_i v_i$, which converges absolutely. If E is any subset of $\mathbf{K}^I$, we denote $E^\perp$ the set of all the elements of $\mathbf{K}^I$ which are in duality with all the elements of E. This set $E^\perp$ (the *dual* of E) is obviously a sequence space. If $E \subseteq \mathbf{K}^I$ satisfies $E = E^{\perp\perp}$, then E is a Hausdorff locally convex topological vector space (see [Jar81, Sch73, Köt66]), with topology defined by the following semi-norms $(N_{x'})_{x'\in E^\perp}$:

$$N_{x'}(x) = \sum_{i\in I} |x'_i x_i|$$

Since this collection $\mathcal{N}$ of semi-norms is directed (that is, if $N, N' \in \mathcal{N}$ there exists $N'' \in \mathcal{N}$ such that $N(x), N'(x) \leq N''(x)$ for all x) and satisfies also, for each $\lambda \in \mathbf{R}^+$, $N \in \mathcal{N} \Rightarrow \lambda N \in \mathcal{N}$, the induced locally convex topology can be described as follows: a subset U of E is a neighborhood of 0 in E iff there is an element x' of $E^\perp$ such that

$$\forall x \in E \quad N_{x'}(x) < 1 \Rightarrow x \in U\,.$$

When $E \subseteq \mathbf{K}^I$ satisfies $E^{\perp\perp} = E$, we denote by $\varepsilon(E)$ the locally convex topology just described above. Unless otherwise specified, such a subspace E of $\mathbf{K}^I$ will always be considered as equipped with the topology $\varepsilon(E)$. If E is any subset of $\mathbf{K}^I$, then $E^{\perp\perp\perp} = E^\perp$, and so the dual E of any subset of $\mathbf{K}^I$ is a subset of $\mathbf{K}^I$ which satisfies $E^{\perp\perp} = E$.

Remark: This topology $\varepsilon(E)$ is in general strictly finer than the *weak* topology on E and strictly coarser than the *strong* topology, and is called the *normal* topology of the sequence space E in [Sch73, Köt66]. In these books, a sequence space E such that $E = E^{\perp\perp}$, equipped with its normal topology, is called a *perfect* sequence space (*vollkommene Folgenraum*). The weak topology is the locally convex topology on E associated to the semi-norms

$$\nu_F : x \mapsto \sup_{x'\in F} |\langle x, x'\rangle|$$

where the sets F are arbitrary finite subsets of $E^\perp$, whereas the strong topology, which is also locally convex, is defined in the same way, but now with F belonging to the much larger class of all *bounded* subsets of $E^\perp$ (F is bounded when $N_x(F)$ is bounded in $\mathbf{R}^+$, for each $x \in E$). See [Jar81, Sch73] for more details on these notions.

If $x \in \mathbf{K}^I$, we denote by $|x|$ the family $(|x_i|)_{i\in I} \in (\mathbf{R}^+)^I \subseteq \mathbf{K}^I$. The next property (sometimes called *solidity* in the literature on sequence spaces) is obvious in the present setting, but will be intensively used in the sequel.

Lemma 1 *If $E \subseteq \mathbf{K}^I$ satisfies $E^{\perp\perp} = E$, then*

$$x \in E \Leftrightarrow \exists y \in E \ \ |x| \leq |y|\ .$$

The next lemma (as part of the material in this section) is completely standard, we give its proof because it is simple.

Lemma 2 *Let $E \subseteq \mathbf{K}^I$ be such that $E = E^{\perp\perp}$. Then E is complete.*

Proof: This actually results from the completeness of the Banach space $\ell^1(I)$. We give a direct proof. Let $(x(\gamma))_{\gamma\in\Gamma}$ be a Cauchy net in E (since I is at most denumerable, we could as well restrict our attention to sequences). So $(\Gamma, \leq)$ is a directed set and for all $x' \in E^\perp$ there exists $\gamma \in \Gamma$ such that, for $\delta, \delta' \geq \gamma$, one has $N_{x'}(x(\delta) - x(\delta')) < 1$. Then for each $i \in I$, the net $(x(\gamma)_i)_{\gamma\in\Gamma}$ is Cauchy in $\mathbf{K}$ and hence has a limit x_i. We show first that the family $x = (x_i)_{i\in I}$ belongs to E. So let $x' \in E^\perp$, we show that $N_{x'}(x) < \infty$. For each $n \in \mathbf{N}$, choose $\gamma_n \in \Gamma$ such that $N_{x'}(x(\delta) - x(\delta')) < 2^{-n}$ for $\delta, \delta' \geq \gamma_n$. We may assume the sequence (γ_n) to be monotone, and we set $x(n) = x(\gamma_n)$. For each $i \in I$, the sequence $(x'_i x(n)_i)_{n\in\mathbf{N}}$ clearly converges to $x'_i x_i$ in $\mathbf{K}$. Let $J \subseteq I$ be finite, we have

$$\begin{aligned}
\sum_{i\in J} |x'_i x_i| &= \sum_{i\in J} \left| x'_i x(0)_i + \sum_{n=0}^{\infty} \left(x'_i x(n+1)_i - x'_i x(n)_i \right) \right| \\
&\leq \sum_{i\in J} |x'_i x(0)_i| + \sum_{i\in J} \sum_{n=0}^{\infty} |x'_i x(n+1)_i - x'_i x(n)_i| \\
&= \sum_{i\in J} |x'_i x(0)_i| + \sum_{n=0}^{\infty} \sum_{i\in J} |x'_i x(n+1)_i - x'_i x(n)_i| \\
&\leq N_{x'}(x(0)) + \sum_{n=0}^{\infty} N_{x'}(x(n+1) - x(n)) \leq N_{x'}(x(0)) + 2
\end{aligned}$$

and we conclude since this holds for all finite $J \subseteq I$. One shows exactly in the same way that $N_{x'}(x - x(n)) \leq 2^{-n}$, and from this it follows that the net (x_γ) converges to x in E. ∎

We denote by $\mathbf{K}^{(I)}$ the subspace of $\mathbf{K}^I$ whose elements are the families which vanish almost everywhere.

Lemma 3 *Let $E \subseteq \mathbf{K}^I$ be such that $E = E^{\perp\perp}$. Then $\mathbf{K}^{(I)}$ is dense in E. Therefore, E is always separable (that is, contains a dense countable subset).*

So E can also be considered as the completion of $\mathbf{K}^{(I)}$ for the topology $\varepsilon(E)$. The proof is trivial.

Remark: Such a space E satisfying $E^{\perp\perp} = E$ is not necessarily a Banach space. Assume that I is infinite and take for E the space ℓ^1 of all the $x \in \mathbf{K}^I$ such that $\sum_{i\in I} |x_i|$ converges. Then $E^\perp$ is the space of all bounded families. Indeed, if $x' \in \mathbf{K}^I$ is unbounded, one can find a sequence $(i_n)_{n\in\mathbf{N}}$ of pairwise distinct elements of I such that $\left|x'_{i_n}\right| \geq 2^n$. If we define $x \in \mathbf{K}^I$ by $x_i = 0$ if $i \notin \{i_n \mid n \in \mathbf{N}\}$ and $x_i = 2^{-n}$ if $i = i_n$, then $x \in E$ and x and x' are not in duality, so $x' \notin E^\perp$. Therefore $E = E^{\perp\perp}$, and as a topological vector space, E is easily seen to be the usual Banach space ℓ^1. But $E^\perp$ is identical to ℓ^∞ only as a vector space; its topology (induced by E) is strictly coarser than the topology of the Banach space ℓ^∞. Indeed, if $x \in E$, there is $i \in I$ such that $|x_i| \leq 1/4$ and for such an i, we have $N_x(x') = 1/2$ for $x' \in E^\perp$ defined by $x'_j = 0$ if $j \neq i$ and $x'_i = 2$. So the unit ball of the semi-norm N_x is not contained in the unit ball of the norm of ℓ^∞, and therefore, this latter unit ball is not a neighborhood of 0 in $E^\perp$. Observe by the way that $E^\perp$ is separable, whereas ℓ^∞ is not.

More generally (see [Sch73, Köt66]), given $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, one can establish a linear isomorphism between $(\ell^p)^\perp$ and ℓ^q, and therefore $(\ell^p)^{\perp\perp} = \ell^p$. When $p > 1$, neither ℓ^p nor ℓ^q can be a Banach space, when equipped with their topologies $\varepsilon(\ell^p)$ and $\varepsilon(\ell^q)$ (it is proved in [Jar81] that any Köthe sequence space – see below the definition of this notion – which is a Banach space is linearly homeomorphic to ℓ^1). An easy direct reasoning (similar to the one above in the case ℓ^1/ℓ^∞) shows that indeed the topology $\varepsilon(\ell^p)$ is strictly coarser than the topology of the Banach space ℓ^p.

The duality between E and $E^\perp$ is determined by the following bilinear map, which is well-defined by definition (the sum involved in this definition converges absolutely in $\mathbf{K}$).

$$\begin{array}{rcl} E \times E^\perp & \to & \mathbf{K} \\ (x, x') & \mapsto & \langle x, x' \rangle = \sum_{i \in I} x_i x'_i \end{array}$$

Remark: This map is separately continuous, but not continuous at $(0,0)$ (for the product topology), as soon as the set I is infinite. Indeed, if it were continuous at $(0,0)$, there would exist $y' \in E^\perp$ and $y \in E$ such that, whenever $(x, x') \in E \times E^\perp$ satisfy $N_{y'}(x) < 1$ and $N_y(x') < 1$ then $|\langle x, x' \rangle| < 1$. But since $\sum_{i \in I} |y_i y'_i| < \infty$, there exists $i \in I$ such that $|y_i y'_i| < \frac{1}{8}$. Then one can find $x, x' \in (\mathbf{R}^+)^I$ such that $x_j = x'_j = 0$ for $j \neq i$ and $x_i \leq \frac{1}{2|y'_i|}$, $x'_i \leq \frac{1}{2|y_i|}$ and $x_i x'_i \geq 2$, and we have a contradiction.

Again, the following simple lemma is completely standard.

Lemma 4 *A semi-norm $N : E \to \mathbf{R}^+$ is continuous iff there exists $x' \in E^\perp$ such that for all $x \in E$, $N(x) \leq N_{x'}(x)$.*

Proof: Assume first that N is continuous. There is a neighborhood V of 0 in E such that $x \in V \Rightarrow N(x) < 1$. So, by the very definition of the topology of E, there exists $x' \in E^\perp$ such that $N_{x'}(x) < 1 \Rightarrow N(x) < 1$, for all $x \in E$. Assume that, for some $x \in E$, $N(x) > N_{x'}(x)$. Then there exists $t > 0$ such that $N(tx) \geq 1 > N_{x'}(tx)$ and we have a contradiction.

Conversely, let N be a semi-norm on E and let $x' \in E^\perp$ be such that $N \leq N_{x'}$. Let $x \in E$ and let $\varepsilon > 0$. Let U be the open unit ball of the semi-norm $N_{\frac{1}{\varepsilon}x'}$ and let $y \in E$ be such that $y - x \in U$. We have $N(y) = N(y - x + x) \leq N(y - x) + N(x)$ and so $|N(y) - N(x)| \leq N(y - x) \leq N_{x'}(y - x) < \varepsilon$. So N is continuous. ∎

Since in $\mathbf{K}$ there is a basis of neighborhood of 0 consisting only of disks, we have:

Lemma 5 *A linear form $f : E \to \mathbf{K}$ is continuous iff the semi-norm $|f| : E \to \mathbf{R}^+$ which maps x to $|f(x)|$ is continuous.*

For $i \in I$, we denote by e_i the element of $\mathbf{K}^I$ defined by $(e_i)_j = \delta_{i,j}$ (the Kronecker symbol). These vectors constitute the "canonical absolute basis" of E.

Lemma 6 *If $E \subseteq \mathbf{K}^I$ satisfies $E^{\perp\perp} = E$, then $E^\perp$ is linearly isomorphic to the topological dual E' of E.*

Proof: First, if $x' \in E^\perp$, then the map $x \mapsto \langle x, x' \rangle$ is well defined and linear from E to $\mathbf{K}$. It is continuous by lemma 4 since $|\langle x, x' \rangle| \leq N_{x'}(x)$. We denote by $\widehat{x'}$ this continuous linear map. Clearly, the map $x' \mapsto \widehat{x'}$ is linear. Conversely, given a continuous linear map $f : E \to \mathbf{K}$, the semi-norm $|f|$ is continuous and so there is $x' \in E^\perp$ such that $|f|(x) \leq N_{x'}(x)$ for all $x \in E$. Taking $x = e_i$, we get $|f(e_i)| \leq |x'_i|$. From this, and from lemma 1, it results that setting $y'_i = f(e_i)$ for each $i \in I$, one defines an element y' of $E^\perp$ that we denote by $\mathsf{M}(f)$. The map $f \mapsto \mathsf{M}(f)$ is clearly linear. Lemma 3, together with the continuity and linearity of f, implies that $f(x) = \langle x, \mathsf{M}(f)\rangle$ for each $x \in E$, that is $\widehat{\mathsf{M}(f)} = f$. Conversely, for $x' \in E^\perp$, one has for each $i \in I$: $\mathsf{M}(\widehat{x'})_i = \widehat{x'}(e_i) = \langle e_i, x'\rangle = x'_i$ and the announced linear bijective correspondence is established. ∎

2.1 Köthe sequence spaces

In [Jar81], particular subspaces E of $\mathbf{K}^I$ are obtained as the duals of *Köthe sets*, that is, subsets P of $(\mathbf{R}^+)^I$ such that

- P is directed for the pointwise order on functions, that is: $\forall p, q \in P\, \exists r \in P\; p, q \leq r$
- and $\forall i \in I\, \exists p \in P\; p_i \neq 0$.

In that situation, one writes traditionally $E = \Lambda(P)$ and E is called a *Köthe sequence space*; $\Lambda(P)$ is always considered as a topological vector space for the topology defined by the semi-norms CN_p (for $p \in P$ and $C \in \mathbf{R}^+$). Indeed, the first condition on P ensures that the absolutely convex sets $\{x \in \Lambda(P) \mid N_p(x) < \varepsilon\}$ constitute a filter basis at 0, and the second condition ensures that the corresponding topology is Hausdorff.

Lemma 7 *The topologies induced on $P^\perp$ by P and by $P^{\perp\perp}$ are identical if and only if for any $x \in P^{\perp\perp}$ there exists $p \in P$ and $C \in \mathbf{R}^+$ such that $|x| \leq Cp$.*

Proof: First, since $P \subseteq P^{\perp\perp}$, the topology τ induced by $P^{\perp\perp}$ is always finer than the topology τ' induced by P. If the condition stated in the lemma holds, it is clear that τ' is finer than τ. Conversely, assume that τ' is finer than τ. Then for any $x \in P^{\perp\perp}$, the semi-norm N_x is τ'-continuous, since it is τ-continuous, hence there exists $p \in P$ and $C \in \mathbf{R}^+$ such that $CN_p \geq N_x$ (see the proof of lemma 4), but this in turn implies $Cp \geq |x|$. ∎

Consequently, the topology induced by $P^{\perp\perp}$ can be strictly finer than the topology induced by P. Consider indeed the following situation: let $\mathcal{D}$ be a maximal ideal of $\mathcal{P}(\mathbf{N})$, that is a subset of $\mathcal{P}(\mathbf{N})$ which is closed under finite unions, downwards closed with respect to inclusion, does not contain $\mathbf{N}$ and is maximal with all these properties. Assume moreover that all the finite subsets of $\mathbf{N}$ belong do $\mathcal{D}$; such a maximal ideal exists by Zorn's lemma. Then let P be the set of all characteristic maps of elements of $\mathcal{D}$, it is a Köthe set. The element u of $\mathbf{K}^\mathbf{N}$ which is constantly equal to 1 belongs to $P^{\perp\perp}$, and so by lemma 7 the topology induced by $P^{\perp\perp}$ on $P^\perp$ is strictly finer than the topology induced by P. That $u \in P^{\perp\perp}$ results from the fact that $P^\perp = \ell^1$ (as a vector space) as, if this were not the case, there would exist $p \in \mathbf{K}^\mathbf{N}$ such that $p \notin \ell^1$ and $p \in P^\perp$. But for such a p, it is possible to find two disjoint infinite subsets of $\mathbf{N}$, L and R, such that none of the restrictions of p to L and R belong to ℓ^1. Since $\mathcal{D}$ is maximal, we must have $L \in \mathcal{D}$ or $\mathbf{N} \setminus L \in \mathcal{D}$, and hence $L \in \mathcal{D}$ or $R \in \mathcal{D}$, in contradiction with the fact that $p \in P^\perp$.

2.2 Köthe spaces and linear continuous maps

We adopt the following definition of Köthe space, which coincides with the notion of perfect (vollkommene) sequence space of [Sch73, Köt66]. The discussion above shows that it is more restrictive than the notion of Köthe sequence spaces as defined in [Jar81].

Definition 8 A Köthe space is a pair $X = (|X|, E_X)$ where $|X|$ is an at most denumerable set (the *web*) and $E_X \subseteq \mathbf{K}^{|X|}$ satisfies $E_X{}^{\perp\perp} = E_X$. If X is a Köthe space, its *dual* $X^\perp$ is defined by $|X^\perp| = |X|$ and $E_{X^\perp} = E_X{}^\perp$. When X is a Köthe space, E_X will always be considered as topological vector space, with topology $\varepsilon(E_X)$ (the normal topology).

Observe that when $|X|$ is finite, $E_X = \mathbf{K}^{|X|}$, with the usual product topology.

Lemma 9 *A linear function $f : E_X \to E_Y$ is continuous iff for all $y' \in E_{Y^\perp}$, the semi-norm $N_{y'} \circ f : E_X \to \mathbf{R}^+$ is continuous.*

Proof: Assume that for all $y' \in E_{Y^\perp}$ the semi-norm $N_{y'} \circ f$ is continuous. It suffices to show that f is continuous at 0. Let V be a neighborhood of 0 in E_Y. Let $y' \in E_{Y^\perp}$ be such that the open unit ball of the semi-norm $N_{y'}$ is included in V. Since the semi-norm $N_{y'} \circ f$ is continuous, there exists $x' \in E_{X^\perp}$ such that $N_{y'} \circ f \leq N_{x'}$. Now if $x \in E_X$ satisfies $N_{x'}(x) < 1$, we have $f(x) \in V$ and so $f^{-1}(V)$ is a 0-neighborhood in E_X, so f is continuous. The converse implication is trivial. ∎

Let $f : E_X \to E_Y$ be a linear function. We define its *matrix* $\mathsf{M}(f) \in \mathbf{K}^{|X|\times|Y|}$ by $\mathsf{M}(f)_{a,b} = f(e_a)_b$.

Lemma 10 *Let $f : E_X \to E_Y$ be linear. If f is continuous, then for all $x \in E_X$ and $y' \in E_{Y^\perp}$, the double sum*

$$\sum_{a\in|X|,b\in|Y|} \mathsf{M}(f)_{a,b} x_a y'_b$$

converges absolutely.

Proof: Let $y' \in E_{Y^\perp}$; the semi-norm $N_{y'} \circ f$ on E_X being continuous, there exists $x' \in E_{X^\perp}$ such that

$$\forall x \in E_X \quad N_{y'}(f(x)) \leq N_{x'}(x)\,,$$

so that in particular (taking $x = e_a$)

$$\forall a \in |X| \quad \sum_{b\in|Y|} \left|y'_b \mathsf{M}(f)_{a,b}\right| \leq \left|x'_a\right| \,. \tag{1}$$

Now let $x \in E_X$, the sum $\sum_{a\in|X|} |x_a x'_a|$ converges, and so by (1) the sum

$$\sum_{a\in|X|} |x_a| \left(\sum_{b\in|Y|} \left|y'_b \mathsf{M}(f)_{a,b}\right|\right)$$

converges and the lemma is proved, since we are dealing with sums of positive terms. ∎

Lemma 11 *Let $M \in \mathbf{K}^{|X|\times|Y|}$ be a matrix such that the sum*

$$\sum_{a\in|X|,b\in|Y|} M_{a,b} x_a y'_b$$

converges absolutely for all $x \in E_X$ and $y' \in E_{Y^\perp}$. Then the linear function $f : E_X \to \mathbf{K}^{|Y|}$ given by $f(x)_b = \sum_{a\in|X|} M_{a,b} x_a$ is well defined, takes its values in E_Y and is continuous from E_X to E_Y.

Proof: By our assumption about M, taking $y' = e_b$, we see that $(M_{a,b})_{a\in|X|} \in E_{X^\perp}$ and so f is well defined and is obviously linear $E_X \to \mathbf{K}^{|Y|}$. Moreover, if $x \in E_X$, then $f(x) \in E_Y$ as indeed, if $y' \in E_{Y^\perp}$, one has

$$\begin{aligned} N_{y'}(f(x)) &= \sum_{b\in|Y|} |y'_b| \left| \sum_{a\in|X|} M_{a,b}x_a \right| \\ &\leq \sum_{b\in|Y|} |y'_b| \sum_{a\in|X|} |M_{a,b}x_a| < \infty \end{aligned}$$

by our assumption about M.

It remains to show that f is continuous. Let $y' \in E_{Y^\perp}$ and let $x' \in E_{X^\perp}$ be given by $x'_a = \sum_{b\in|Y|} |M_{a,b}y'_b|$ (that x' is well defined and belongs to $E_{X^\perp}$ is a consequence of our assumption about M). Then if $x \in E_X$ we have $N_{y'}(f(x)) \leq N_{x'}(x)$ and we conclude that f is continuous by lemma 9. ∎

We denote by $\widehat{M}$ the function f defined above, and when $x \in E_X$, we denote sometimes by $M \cdot x$ the vector $\widehat{M}(x) \in E_Y$.

Given two Köthe spaces X and Y, the set E of all matrices $M \in \mathbf{K}^{|X|\times|Y|}$ such that the sum

$$\sum_{a\in|X|,b\in|Y|} M_{a,b}x_a y'_b$$

converges absolutely for all $x \in E_X$ and $y' \in E_{Y^\perp}$ is a subset of $\mathbf{K}^{|X|\times|Y|}$ which satisfies $E^{\perp\perp} = E$ (because it is defined as the dual of something), and so the pair $(|X| \times |Y|, E)$ is a Köthe space that we denote by $X \multimap Y$. If S and T are topological vector spaces, we denote by $\mathcal{L}(S,T)$ the vector space of all linear continuous maps from S to T.

Proposition 12 *The linear maps $M \mapsto \widehat{M}$ from $E_{X\multimap Y}$ to $\mathcal{L}(E_X, E_Y)$ and $f \mapsto \mathsf{M}(f)$ from $\mathcal{L}(E_X, E_Y)$ to $E_{X\multimap Y}$ define a linear isomorphism between these two spaces.*

Proof: Given $M \in E_{X\multimap Y}$, we have $\mathsf{M}(\widehat{M})_{a,b} = \widehat{M}(e_a)_b = M_{a,b}$ and given $f \in \mathcal{L}(E_X, E_Y)$, for each $b \in |Y|$, the map $g : E_X \to \mathbf{K}$ defined by $g(x) = f(x)_b$ is a continuous linear form, and so $g = \widehat{\mathsf{M}(g)}$ by lemma 6. But $\mathsf{M}(g)_a = g(e_a) = f(e_a)_b = \mathsf{M}(f)_{a,b}$ and so we have, for each $x \in E_X$, $f(x)_b = \sum_{a\in|X|} \mathsf{M}(f)_{a,b}x_a$, and since this holds for each $b \in |Y|$, finally we get that $f = \widehat{\mathsf{M}(f)}$. ∎

We have defined a category $\mathcal{K}_\mathbf{K}$ whose objects are the Köthe spaces and whose morphisms are the linear continuous functions between them.

The identity morphism from X to X is of course the usual identity function, and its matrix I is the diagonal matrix $I_{a,b} = \delta_{a,b}$. Corresponding to composition of functions, the product of matrices is defined as usual: let $A \in E_{X\multimap Y}$ and $B \in E_{Y\multimap Z}$, the matrix $BA \in E_{X\multimap Z}$ is given by

$$(BA)_{a,c} = \sum_{b\in|Y|} A_{a,b}B_{b,c}\,.$$

This sum converges absolutely because $(A_{a,b})_{b\in|Y|} \in E_Y$ as $e_a \in E_X$ and $(B_{b,c})_{b\in|Y|} \in E_{Y^\perp}$ as $e_c \in E_{Z^\perp}$. Moreover, if $f : E_X \to E_Y$ and $g : E_Y \to E_Z$ are the linear continuous maps defined by A and B respectively, then $\mathsf{M}(g \circ f)_{a,c} = g(f(e_a))_c = g(\sum_{b\in|Y|} A_{a,b}e_b)_c = (BA)_{a,c}$ by linearity and continuity of g. It follows, since $g \circ f$ is continuous, that $BA \in E_{X\multimap Z}$, and we have seen that $\mathsf{M}(g \circ f) = BA$.

An obvious consequence of the considerations above on continuous linear maps is the following

Proposition 13 *Let $f : E_X \to E_Y$ be linear and continuous. Then its transpose ${}^{\mathrm{t}}f : E'_Y \to E'_X$ (given by ${}^{\mathrm{t}}f(v') = v' \circ f$) defines a continuous linear map $f^\perp : E_{Y^\perp} \to E_{X^\perp}$ whose matrix is given as usual by $\mathsf{M}({}^{\mathrm{t}}f) = \mathsf{M}(f)^\perp$ where $A^\perp_{b,a} = A_{a,b}$ for a matrix $A \in \mathbf{K}^{|X|\times|Y|}$.*

Proof: The function $f^\perp$ is given by $f^\perp(y') = \mathsf{M}(\widehat{y'} \circ f)$. It is clearly well defined, linear, and takes its values in $E_{X^\perp}$. Let $A = \mathsf{M}(f) \in E_{X\multimap Y}$. Since clearly $A^\perp \in E_{Y^\perp \multimap X^\perp}$, it suffices to show that $f^\perp = \widehat{A^\perp}$. So let $y' \in E_{Y^\perp}$ and let $a \in |X|$, we have

$$\begin{aligned} f^\perp(y')_a &= \mathsf{M}(\widehat{y'} \circ f)_a \\ &= \widehat{y'}(f(e_a)) \\ &= \langle y', f(e_a)\rangle \\ &= \sum_{b\in|Y|} A_{a,b} y'_b \\ &= \widehat{A^\perp}(y')_a \,. \end{aligned}$$

∎

The map $f^\perp$ is characterized by the following standard adjunction property:

$$\forall x \in E_X\, \forall y' \in E_{Y^\perp} \quad \langle f(x), y'\rangle = \langle x, f^\perp(y')\rangle \,.$$

One defines in that way a contravariant involutive endofunctor $X \mapsto X^\perp$ on $\mathcal{K}_{\mathbf{K}}$.

Let X and Y be Köthe spaces and let $\varphi : |X| \to |Y|$ be a bijection. We denote by φ^* the reindexing map $\mathbf{K}^{|Y|} \to \mathbf{K}^{|X|}$ given by $\varphi^*(y)_a = y_{\varphi(a)}$. The following lemma is easy but will be useful in the sequel.

Lemma 14 *If, for each $y \in \mathbf{K}^{|Y|}$, one has $y \in E_Y$ iff $\varphi^*(y) \in E_X$, then φ^* is a linear homeomorphism from E_Y to E_X.*

Proof: It is clear that φ^* is linear, and since φ^{-1} satisfies the same condition as φ, it will be sufficient to show that φ^* is continuous. So let $x' \in E_{X^\perp}$ and let $y' = (\varphi^{-1})^*(x')$. We show that $y' \in E_{Y^\perp}$, so let $y \in E_Y$. We know that $\varphi^*(y) \in E_X$, so the sum $\sum_{a\in|X|} \left|y_{\varphi(a)} x'_a\right|$ converges, that is, the sum $\sum_{b\in|Y|} \left|y_b x'_{\varphi^{-1}(b)}\right|$ converges (to the same value). Since this holds for each $y \in E_Y$, we have $y' \in E_{Y^\perp}$. We have also shown that for each $y \in E_Y$ we have $N_{x'}(\varphi^*(y)) = N_{y'}(y)$, and so φ^* is continuous. ∎

Of course, not all linear homeomorphisms between Köthe spaces can be described as bijections between the webs (in sharp contrast with what happens in the "discrete" denotational semantics), but the "logical" isomorphisms which will be used for describing the categorical structures of the model under study will be of that particular shape.

2.3 Equicontinuous sets

We start a short digression about the notion of equicontinuous sets. These sets indeed can easily be used for describing the topology of the dual of a Köthe space, but unfortunately, this description does not seem to extend simply to more general function spaces, see section 2.6.

If X and Y are Köthe spaces, a subset H of $\mathcal{L}(E_X, E_Y)$ is *equicontinuous* if, for any 0-neighborhood V in E_Y, the set

$$\bigcap_{f\in H} f^{-1}(V)$$

is a 0-neighborhood in E_X. If $H \subseteq E_X$, we shall say that H is equicontinuous if the set $\widehat{H} = \{\widehat{x} \mid x \in H\} \subseteq E'_{X^\perp}$ is an equicontinuous set of continuous linear forms on $E_{X^\perp}$.

Lemma 15 *A subset H of E_X is equicontinuous iff there exists $x \in E_X$ such that for all $y \in H$, $|y| \leq |x|$.*

Proof: Indeed, H is equicontinuous, by definition, iff there is a neighborhood V' of 0 in $E_{X^\perp}$ such that

$$\forall x' \in V' \, \forall y \in H \quad \left|\langle y, x'\rangle\right| < 1$$

but this in turn is equivalent to requiring that there exists $x \in E_X$ such that for all $x' \in E_{X^\perp}$ and all $y \in H$, if $N_x(x') < 1$ then $|\langle y, x'\rangle| < 1$. Using the same kind of trick as in the proof of lemma 4, one shows that this is equivalent to

$$\forall x' \in E_{X^\perp} \, \forall y \in H \quad \left|\langle y, x'\rangle\right| \leq N_x(x')$$

which clearly implies (taking $x' = e_a$) that $|y_a| \leq |x_a|$ for all $a \in |X|$ and all $y \in H$. It is obvious that conversely, if $|y| \leq |x|$, then the condition above is fulfilled. ∎

Given a Köthe space X, we denote by $\mathcal{E}_X$ the set of all the equicontinuous subsets of E_X; it is clear that each of these sets is bounded (that is, for each $B \in \mathcal{E}_X$ and each continuous semi-norm p, the set $p(B)$ is bounded). We show that these sets are even relatively compact.

Lemma 16 *Let $x \in E_X$. Then $R(x) = \{y \in E_X \mid |y| \leq |x|\}$ is a compact subset of E_X.*

Proof: Let $\mathbf{D}$ be the closed unit disk of $\mathbf{K}$. Let $\varphi : \mathbf{D}^{|X|} \to E_X$ be the function which maps $t \in \mathbf{D}^{|X|}$ to $tx = (t_a x_a)_{a \in |X|}$ which belongs to E_X because the family t is bounded. It is clear that $\varphi(\mathbf{D}^{|X|}) = R(x)$, and since $\mathbf{D}^{|X|}$ is a compact space (for the product topology), it will be sufficient to show that φ is continuous.

So let $t \in \mathbf{D}^{|X|}$ and let $x' \in E_{X^\perp}$. Since the sum $\sum_{a \in |X|} |x_a x'_a|$ converges to, say, $S \in \mathbf{R}^+$, there is a finite subset A of $|X|$ such that $\sum_{a \notin A} |x_a x'_a| \leq 1/4$. Let U be the set of all the elements s of $\mathbf{D}^{|X|}$ such that

$$\forall a \in A \quad |s_a - t_a| < \varepsilon = \frac{1}{4(1+S)},$$

this is a neighborhood of t in $\mathbf{D}^{|X|}$ (for the product topology) because A is finite. If $s \in U$, we have

$$\begin{aligned} N_{x'}(\varphi(s) - \varphi(t)) &= \sum_{a \notin A} \left|x'_a x_a (s_a - t_a)\right| + \sum_{a \in A} \left|x'_a x_a (s_a - t_a)\right| \\ &\leq 2 \sum_{a \notin A} \left|x'_a x_a\right| + S\varepsilon < 1 \end{aligned}$$

by definition of ε and so φ is continuous. ∎

As we have seen, this implies that each equicontinuous set is relatively compact. The converse does not hold in general. Consider the space $\ell^1(\mathbf{N}^+)$, with canonical basis $(e_n)_{n \in \mathbf{N}^+}$. Then the set

$$\{0\} \cup \{\frac{1}{n} e_n \mid n \in \mathbf{N}^+\}$$

is compact since each neighborhood of 0 in $\ell^1(\mathbf{N}^+)$ contains almost all the vectors $\frac{1}{n} e_n$ (remember that $\ell^1(\mathbf{N}^+)$ is simply the usual Banach space). But clearly, there is no $x \in \ell^1(\mathbf{N}^+)$ such that $\frac{1}{n} e_n \in R(x)$ for all $n \in \mathbf{N}^+$.

Equicontinuous sets provide a simple characterization of the topology of the dual of a Köthe space.

Proposition 17 *Let X be a Köthe space. A subset U' of $E_{X^\perp}$ is a 0-neighborhood in $E_{X^\perp}$ iff there is an equicontinuous subset H of E_X such that, for all $x' \in E_{X^\perp}$, if $|\langle x, x'\rangle| < 1$ for all $x \in H$, then $x' \in U'$. In other words, the topology of $E_{X^\perp}$ is the topology of uniform convergence on the equicontinuous subsets of E_X.*

This is an obvious consequence of lemma 15.

Lemma 18 *Let X and Y be Köthe spaces and let $f : E_Y \to E_X$ be a continuous linear map. If $H \in \mathcal{E}_Y$ then $f(H) \in \mathcal{E}_X$.*

Proof: Let $A = |\mathsf{M}(f)| \in E_{X \multimap Y}$ and let $x \in E_X$ be positive (that is, $x_a \geq 0$ for each $a \in |X|$) and such that $H \subseteq R(x)$. Then $f(H) \subseteq R(Ax)$ and so $f(H) \in \mathcal{E}_Y$. ∎

2.4 Direct sums and products

Let $(X_j)_{j\in J}$ be a countable family of Köthe spaces. Let K be the disjoint sum of the sets $|X_j|$. Let $E \subseteq \mathbf{K}^K$ be defined by: $z \in E$ iff, for all $j \in J$, the restriction $\pi_j(z)$ of the family z to $|X_j| \subseteq K$ belongs to E_{X_j}. Then $z' \in E^\perp$ iff $\pi_j(z') \in E_{X_j{}^\perp}$ for each $j \in J$ and, moreover, $\pi_j(z') = 0$ for almost all $j \in J$. Therefore, (K, E) is a Köthe space that we denote by $\&_{j\in J}\, X_j$. The dual operation $\oplus_{j\in J} X_j = (\&_{j\in J}\, X_j{}^\perp)^\perp$ can be described as follows: $|\oplus_{j\in J} X_j|$ is the disjoint sum of the $|X_j|$'s, and $z \in E_{\oplus_{j\in J} X_j}$ iff $z \in E_{\&_{j\in J} X_j}$ and $\pi_j(z) = 0$ for almost all $j \in J$. It is clear then that $\&_{j\in J}\, X_j$ is the cartesian product of the spaces X_j in the category $\mathcal{K}_\mathbf{K}$, and that $\oplus_{j\in J} X_j$ is their direct sum. Given $x = (x(j))_{j\in J} \in E_{\&_{j\in J} X_j}$ and $x' = (x'(j))_{j\in J} \in E_{\oplus_{j\in J} X_j{}^\perp}$, we have

$$\langle x, x'\rangle = \sum_{j\in J} \langle x(j), x'(j)\rangle \quad \text{and} \quad N_{x'}(x) = \sum_{j\in J} N_{x'(j)}(x(j))$$

which are finite sums.

So the space $E_{\&_{j\in J} X_j}$ is canonically isomorphic to $\prod_{j\in J} E_{X_j}$, and its topology is the product topology. Observe also that, as usual in vector spaces, the finite direct sum $E_{X\oplus Y}$ and the finite direct product $E_{X\& Y}$ are identical.

In particular, $\mathbf{K}^\mathbf{N}$ is a Köthe space, with dual $\mathbf{K}^{(\mathbf{N})}$, the space of almost everywhere vanishing sequences. As the infinite denumerable sum of 1 (the neutral element of the tensor product, see section 2.5) in the category $\mathcal{K}_\mathbf{K}$, this latter space will correspond to the domain of "flat natural numbers". The space $\mathbf{K}^{(\mathbf{N})}$ is not metrizable (we shall prove this fact, using a simple Cantor diagonalization argument), which shows that we cannot restrict our attention to Frechet Köthe spaces, for instance (a Frechet space is a complete metrizable lcs). Assume indeed that $\mathbf{K}^{(\mathbf{N})}$ is metrizable, then its topology is induced by a denumerable family of neighborhood $(U_n)_{n\in\mathbf{N}}$ of 0. There must exist a family $(x'(n))_{n\in\mathbf{N}}$ of elements of $\mathbf{K}^\mathbf{N}$ such that $N_{x'(n)}(x) < 1 \Rightarrow x \in U_n$ for all $x \in \mathbf{K}^{(\mathbf{N})}$. Clearly, we can assume that $x'(n)$ belong to $(\mathbf{R}^+)^\mathbf{N}$ for each $n \in \mathbf{N}$. Let $x' \in (\mathbf{R}^+)^\mathbf{N}$ be given by $x'_n = x'(n)_n + 1$. Since $(U_n)_{n\in\mathbf{N}}$ generates the topology of $\mathbf{K}^{(\mathbf{N})}$, there must exist $m \in \mathbf{N}$ such that $x \in U_m \Rightarrow N_{x'}(x) < 1$ for all $x \in \mathbf{K}^{(\mathbf{N})}$, and this implies that $x'(m) \geq x'$ (for the product order), which leads to the usual contradiction, spelling out this inequation at index m.

Proposition 19 *Let X be a Köthe space and let $(Y_j)_{j\in J}$ be a family of Köthe spaces. Let $\varphi : |X \multimap \&_{j\in J}\, Y_j| \to |\&_{j\in J}\, (X \multimap Y_j)|$ be the obvious bijection (distributivity of the cartesian product over the disjoint sum). Then φ^* is a linear homeomorphism from $E_{\&_{j\in J}(X\multimap Y_j)}$ to $E_{X\multimap \&_{j\in J} Y_j}$.*

The proof is a straightforward verification using lemma 14. This can also be considered as a piece of abstract non-sense resulting from the fact that & is the cartesian product and from the fact that the functor $Y \mapsto (X \multimap Y)$ has a left adjoint (tensor product, see below).

2.5 Tensor product

One defines of course $X \otimes Y$ by $X \otimes Y = (X \multimap Y^{\perp})^{\perp}$. The dual operation $\parr$ is given by $X \parr Y = X^{\perp} \multimap Y$. In other terms, $X \parr Y$ is the space of all families $P \in \mathbf{K}^{|X|\times|Y|}$ such that the double sum $\sum_{a,b} P_{a,b} x'_a y'_b$ converges absolutely for all $x' \in E_{X^{\perp}}$ and $y' \in E_{Y^{\perp}}$. For $x \in E_X$ and $y \in E_Y$, we define $x \otimes y \in \mathbf{K}^{|X|\times|Y|}$ by $(x \otimes y)_{a,b} = x_a y_b$, and then clearly $E_{X\otimes Y} = \{x \otimes y \mid x \in E_X \text{ and } y \in E_Y\}^{\perp\perp}$.

Lemma 20 *Let $A \in \mathbf{K}^{(|X|\times|Y|)\times|Z|}$. One has $A \in E_{(X\otimes Y)\multimap Z}$ iff for all $x \in E_X$, $y \in E_Y$ and $z' \in E_{Z^{\perp}}$, the sum*

$$\sum_{((a,b),c)\in|(X\otimes Y)\multimap Z|} \left|A_{(a,b),c} x_a y_b z'_c\right|$$

converges.

Proof: The condition is clearly necessary, let us show that it is sufficient, and so assume that it holds. Let $t \in E_{X\otimes Y}$ and $z' \in E_{Z^{\perp}}$. Observe first that for each $(a,b) \in |X| \times |Y|$ the sum $\sum_{c\in|Z|} \left|A_{(a,b),c} z'_c\right|$ converges by our assumption about A. We must show that the sum

$$\sum_{((a,b),c)\in|(X\otimes Y)\multimap Z|} \left|A_{(a,b),c} t_{a,b} z'_c\right| = \sum_{(a,b)\in|X\otimes Y|} |t_{a,b}| \sum_{c\in|Z|} \left|A_{(a,b),c} z'_c\right|$$

converges. But for this, it is sufficient to show that the family $(\sum_{c\in|Z|} \left|A_{(a,b),c} z'_c\right|)_{(a,b)\in|X\otimes Y|}$ belongs to $E_{(X\otimes Y)^{\perp}} = \{x \otimes y \mid x \in E_X \text{ and } y \in E_Y\}^{\perp}$, and this holds iff the sum

$$\sum_{(a,b)\in|X\otimes Y|} |x_a y_b| \sum_{c\in|Z|} \left|A_{(a,b),c} z'_c\right|$$

converges for all $x \in E_X$ and $y \in E_Y$. But this is precisely our hypothesis. ∎

Given matrices $A \in E_{X\multimap Y}$ and $B \in E_{Z\multimap T}$, we define a matrix $A \otimes B = C \in \mathbf{K}^{(|X|\times|Z|)\times(|Y|\times|T|)}$ as follows: $C_{(a,c),(b,d)} = A_{a,b} B_{c,d}$.

Lemma 21 $A \otimes B \in E_{(X\otimes Z)\multimap(Y\otimes T)}$.

Proof: Let $x \in E_X$ and $z \in E_Z$. Let $M \in E_{(Y\otimes T)^{\perp}}$. We assume all these families to be positive. We make the same assumption concerning A and B, since it suffices to show that $|A \otimes B| = |A| \otimes |B| \in E_{(X\otimes Z)\multimap(Y\otimes T)}$. Due to lemma 20, we just have to prove that the sum

$$\sum_{((a,c),(b,d))\in|(X\otimes Z)\multimap(Y\otimes T)|} A_{a,b} B_{c,d} x_a z_c M_{b,d}$$

converges. Denoting by f and g the continuous linear maps defined by A and B, this sum is equal to

$$\sum_{(b,d)\in|Y\otimes T|} M_{b,d} f(x)_b g(z)_d$$

and this latter sum converges since $M \in E_{(Y\otimes T)^{\perp}}$, $f(x) \in E_Y$ and $g(z) \in E_T$. ∎

Given $f \in \mathcal{L}(E_X, E_Y)$ and $g \in \mathcal{L}(E_Z, E_T)$, we denote by $f \otimes g$ the continuous linear function $E_{X\otimes Z} \to E_{Y\otimes T}$ defined by the matrix $\mathsf{M}(f) \otimes \mathsf{M}(g)$.

Then one proves easily the following:

Lemma 22 *If $x \in E_X$ and $z \in E_Z$, then $(f \otimes g)(x \otimes z) = f(x) \otimes g(z)$.*

Due to lemma 3, the finite sums of tensors $x \otimes z$ constitute a dense subspace of $E_{X\otimes Z}$ (which is isomorphic to the algebraic tensor product $E_X \otimes E_Z$), and so this latter property completely determines the map $f \otimes g$. Hence the operation $\otimes$ defined on Köthe spaces is functorial.

We denote by $\otimes$ the bilinear map $E_X \times E_Y \to E_{X\otimes Y}$ which maps (x, y) to $x \otimes y$.

Lemma 23 *The canonical associativity bijection $\varphi : |X \multimap (Y \multimap Z)| \to |(X \otimes Y) \multimap Z|$ induces a linear homeomorphism φ^* from $E_{(X\otimes Y)\multimap Z}$ to $E_{X\multimap(Y\multimap Z)}$.*

Proof: We show that φ satisfies the conditions of lemma 14. So let $M \in \mathbf{K}^{|(X\otimes Y)\multimap Z|}$, and assume first that $M \in E_{(X\otimes Y)\multimap Z}$. Let $x \in E_X$. For each $t \in E_{(Y\multimap Z)^\perp} = E_{Y\otimes Z^\perp}$ we must show that the sum

$$\sum_{(a,b,c)\in|X|\times|Y|\times|Z|} \left|M_{(a,b),c} x_a t_{b,c}\right|$$

is finite, but for this it suffices to show that the family $(\sum_{a\in|X|} M_{(a,b),c} x_a)_{(b,c)}$ (which is well defined by our assumption about M) belongs to $E_{(Y\otimes Z^\perp)^\perp}$. So let $y \in E_Y$ and $z' \in E_{Z^\perp}$, we just have to show that the sum

$$\sum_{(a,b,c)\in|X|\times|Y|\times|Z|} \left|M_{(a,b),c} x_a y_b z'_c\right|$$

is finite, but this directly results from our assumption about M. Conversely, assume that $\varphi^* M \in E_{X\multimap(Y\multimap Z)}$. According to lemma 20, we just have to show that for all $x \in E_X$, $y \in E_Y$ and $z' \in E_{Z^\perp}$, the sum $\sum_{(a,b,c)\in|X|\times|Y|\times|Z|} \left|M_{(a,b),c} x_a y_b z'_c\right|$ is finite, but this results from the fact that $y \otimes z' \in E_{(Y\multimap Z)^\perp}$. ∎

By proposition 12, this linear homeomorphism φ^* induces a (linear) bijection from $\mathcal{L}(E_{X\otimes Y}, E_Z)$ to $\mathcal{L}(E_X, E_{Y\multimap Z})$. If $f : E_{X\otimes Y} \to E_Z$ is linear and continuous, the corresponding linear and continuous function $g : E_X \to (E_{Y\multimap Z})$ maps x to $g(x)$, the matrix of the linear and continuous map $f_x : E_Y \to E_Z$ defined by $f_x(y) = f(x \otimes y)$.

Using lemma 23, one shows easily that $\otimes$ is associative. Symmetry of $\otimes$ and the commutation of the Mac Lane pentagonal diagram are easily checked too. So the category $\mathcal{K}_\mathbf{K}$ is symmetric monoidal closed. It is actually a $\star$-autonomous category in the sense of [Bar79] (see also [Bie95]).

Let $f : E_Y \to E_X$ and $g : E_Z \to E_T$ be linear continuous maps between Köthe spaces. Then we can define a linear continuous map $f \multimap g : E_{X\multimap Z} \to E_{Y\multimap T}$ by its matrix (indexed by $(|X| \times |Z|) \times (|Y| \times |T|)$) as follows: $\mathsf{M}(f \multimap g)_{(a,c),(b,d)} = \mathsf{M}(f)_{b,a}\mathsf{M}(g)_{c,d}$. This matrix belongs to $E_{(X\multimap Z)\multimap(Y\multimap T)}$ by lemma 21, indeed $\mathsf{M}(f \multimap g) = (\mathsf{M}(f) \otimes \mathsf{M}(g)^\perp)^\perp$.

Lemma 24 *The operation $(f, g) \mapsto f \multimap g$ is functorial (contravariant in its first argument and covariant in its second argument). Moreover, if $h \in \mathcal{L}(E_X, E_Z)$, one has $(f \multimap g)(\mathsf{M}(h)) = \mathsf{M}(g \circ h \circ f)$.*

Proof: Functoriality results from the functoriality of the $\otimes$ operation on morphisms. Let $h \in \mathcal{L}(E_X, E_Z)$, we have $(f \multimap g)(\mathsf{M}(h)) = \mathsf{M}(f \multimap g)\mathsf{M}(h)$ (product of matrices) by proposition 12, and an easy computation shows that $\mathsf{M}(f \multimap g)\mathsf{M}(h) = \mathsf{M}(g)\mathsf{M}(h)\mathsf{M}(f)$. ∎

Proposition 25 *The tensor product distributes over direct sums.*

This results from proposition 19.

A central notion in the theory of locally convex spaces is the notion of bounded set (see [Jar81, Sch73]). We conclude this section by observing that the universal bilinear map $\otimes : E_X \times E_Y \to E_{X\otimes Y}$ associated to this tensor product, though not continuous in general, nevertheless preserves bounded sets.

Definition 26 A subset B of a lcs E is *bounded* if, for any neighborhood U of 0 in E, there exists $\lambda \in \mathbf{R}^+$ such that $B \subseteq \lambda U$ or, equivalently, if, for any continuous semi-norm $p : E \to \mathbf{R}^+$, the set $\{p(x) \mid x \in B\}$ is bounded.

Bounded sets are closed under taking subsets and under finite unions. A subset of a direct product of two lcs is bounded iff its two projections are. Moreover, continuous linear maps preserve boundedness (i.e. are bounded). The converse holds for instance in Banach spaces, but is false in general: an lcs E is *bornological* if any bounded linear map from E to a Banach space is continuous. Even in our setting, there are bounded linear forms on $(\ell^1)^\perp$ which are not continuous. Indeed, as a vector space, $(\ell^1)^\perp = \ell^\infty$ and one can check that a subset of $(\ell^1)^\perp$ is bounded iff it is bounded for the norm of the Banach space ℓ^∞. So a bounded linear form on $(\ell^1)^\perp$ is just a continuous linear form on the Banach space ℓ^∞. Then it is a standard exercise to construct, using the Hahn-Banach theorem, a non-zero continuous linear form on ℓ^∞ which maps to 0 all the sequences which converge to 0, and such a linear form cannot be continuous on $(\ell^1)^\perp$; this latter space is not bornological.

The bilinear map $\otimes : E_X \times E_Y \to E_{X\otimes Y}$ cannot be continuous in general, since the bilinear form $E_X \times E_{X^\perp} \to \mathbf{K}$ is not continuous as soon as $|X|$ is infinite, whereas the linear evaluation map $E_{X\otimes X^\perp} \to \mathbf{K}$ is continuous by monoidal closeness. We show that nevertheless this map $\otimes$ is bounded, that is, it maps bounded sets to bounded sets (this condition is certainly not sufficient for a separately continuous bilinear map to factorize through the tensor product).

Assume that $\otimes$ is not bounded. Let B and C be bounded subsets of E_X and E_Y and let $M \in E_{(X\otimes Y)^\perp}$ be such that $\{N_M(x \otimes y) \mid (x, y) \in B \times C\}$ is not bounded. Then we can find in B and C two sequences $(x(n))_{n\in\mathbf{N}}$ and $(y(n))_{n\in\mathbf{N}}$ such that $N_M(x(n) \otimes y(n)) \geq 4^n$ for all $n \in \mathbf{N}$. For each $a \in |X|$, the sequence $(x(n)_a)_{n\in\mathbf{N}}$ is bounded in $\mathbf{K}$ since B is bounded, and so the series $\sum_{n=0}^{\infty} |x(n)_a|\, 2^{-n}$ converges; let us denote by x_a its sum. Then $x = (x_a)_{a\in|X|}$ belongs to E_X: let $x' \in E_{X^\perp}$ be positive, and let A be an upper bound of $\{N_{x'}(z) \mid z \in B\}$, then $N_{x'}(2^{-n}x(n)) \leq 2^{-n}A$ and so $N_{x'}(x) \leq 2A$. Now let $y' \in \mathbf{K}^{|Y|}$ be given by $y'_b = \sum_{a\in|X|} |M_{a,b}|\, x_a$, by our assumption about M, we have $y' \in E_{Y^\perp}$. Therefore, the sequence $N_{y'}(y(n))$ must be bounded. But $N_{y'}(y(n)) = N_M(x \otimes y(n)) \geq 2^{-n} N_M(x(n) \otimes y(n)) \geq 2^n$, contradiction.

2.6 A negative result about the topology of linear function spaces

In view of proposition 17, one might hope to characterize the topology of $E_{X\multimap Y}$ as the topology of uniform convergence on equicontinuous sets. In this section, we provide a counter-example against this conjecture. We take $\mathbf{K} = \mathbf{R}$ and hence $\mathbf{D} = [-1, 1]$.

We start with some simple combinatorial considerations. For $p \in \mathbf{N}$, let $(P_j)_{j=1,\dots,4^p}$ be an enumeration of all the subsets of $\{1, \dots, 2p\}$ (we denote by Q_j the complementary of P_j in $\{1, \dots, 2p\}$) and let $A^{(p)}$ the $\mathbf{N} \times \mathbf{N}$-matrix given by

$$A^{(p)}_{i,j} = \begin{cases} 1 & \text{if } j \leq 4^p \text{ and } i \in P_j \\ -1 & \text{if } j \leq 4^p \text{ and } i \in Q_j \\ 0 & \text{otherwise.} \end{cases}$$

Given $t \in \mathbf{D}^{\mathbf{N}}$, we set

$$S^{(p)}(t) = \sum_{j=1}^{4^p} \left| \sum_{i=1}^{2p} A^{(p)}_{i,j} t_i \right| \in \mathbf{R}^+ .$$

Observe first that, for all $t \in \mathbf{D}^{\mathbf{N}}$, one has $S^{(p)}(t) = S^{(p)}(|t|)$, due to the definition of $A^{(p)}$. Then, using the fact that the function $x \mapsto |a+x| + |a-x|$ is monotone on $\mathbf{R}^+$ (for any given $a \in \mathbf{R}$), we obtain easily

$$\forall t \in \mathbf{D}^{\mathbf{N}} \quad S^{(p)}(t) \leq \sum_{j=1}^{4^p} \left| \sum_{i=1}^{2p} A^{(p)}_{i,j} \right| = T^{(p)} .$$

But we have

$$\begin{aligned} T^{(p)} &= \sum_{j=1}^{4^p} |\#P_j - \#Q_j| \\ &= \sum_{k=0}^{2p} \binom{2p}{k} |k - (2p-k)| \\ &= 4 \sum_{k=0}^{p-1} (p-k) \binom{2p}{k} \end{aligned}$$

Using the identity $k\binom{2p}{k} = (2p-k+1)\binom{2p}{k-1}$, one shows then that $\sum_{k=0}^{p-1}(p-k)\binom{2p}{k} = \frac{1}{2}p\binom{2p}{p}$ and an easy application of Stirling formula shows that $p\binom{2p}{p} \sim C4^p\sqrt{p}$ as p goes to the infinity (for a certain constant $C > 0$). Therefore:

$$\lim_{p\to\infty} \frac{T^{(p)}}{p4^p} = 0 .$$

Using actually only finite-dimensional matrices, we have thus proven the following

Lemma 27 *For all $\varepsilon > 0$, there exists a matrix $A \in \ell^1(\mathbf{N}\times\mathbf{N}) \setminus \{0\}$ such that, for all $t \in \mathbf{D}^{\mathbf{N}}$,*

$$\sum_{j\in\mathbf{N}} \left| \sum_{i\in\mathbf{N}} A_{i,j} t_i \right| \leq \varepsilon \|A\|_1 .$$

Let X and Y be Köthe spaces. The topology of uniform convergence on the equicontinuous subsets of E_X is defined on $E_{X\multimap Y}$ by the following semi-norms

$$\nu_{x,y'}(A) = \sup_{u\in R(x)} N_{y'}(A\cdot u) = \sup_{t\in\mathbf{D}^{|X|}} \sum_{b\in|Y|} \left| \sum_{a\in|X|} A_{a,b} x_a y'_b t_a \right| \leq N_{x\otimes y'}(A) ,$$

for $x \in E_X$ and $y' \in E_{Y^\perp}$. This topology τ is always coarser than the topology of $X \multimap Y$. When Y is finite-dimensional (that is, when E_Y is finite-dimensional, or equivalently when the set $|Y|$ is finite), the two topologies coincide. We show that, in some circumstances, the topology τ is strictly coarser than the topology of $X \multimap Y$.

Let Y be the Köthe space given by $|Y| = \mathbf{N}$ and $E_Y = \ell^1$, and let $X = Y^\perp$, so that E_X is the space of all bounded elements of $\mathbf{K}^{\mathbf{N}}$. Then clearly $E_{X\multimap Y} = \ell^1(\mathbf{N}\times\mathbf{N})$. If the topology of $E_{X\multimap Y}$

were the topology of uniform convergence on the equicontinuous sets, there would exist a constant $k > 0$ and two vectors $x \in E_X$ and $y' \in E_{Y^\perp}$ such that, for all $A \in E_{X \multimap Y}$, $\|A\|_1 \leq k\nu_{x,y'}(A)$. We can assume $\|x\|_\infty, \|y'\|_\infty \leq 1$ (due to the presence in this statement of the multiplicative constant k), and so this statement is equivalent to the existence of $k > 0$ and of $y' \in E_{Y^\perp}$ with $\|y'\|_\infty \leq 1$ such that

$$\forall A \in \ell^1(\mathbf{N} \times \mathbf{N}) \quad \|A\|_1 \leq k \sup_{t \in \mathbf{D}^\mathbf{N}} \sum_{j \in \mathbf{N}} |y'_j| \left| \sum_{i \in \mathbf{N}} A_{i,j} t_i \right|$$

and so finally, if the topology of $E_{X \multimap Y}$ were the topology of uniform convergence on the equicontinuous sets, there would exist a constant $k > 0$ such that

$$\forall A \in \ell^1(\mathbf{N} \times \mathbf{N}) \quad \|A\|_1 \leq k \sup_{t \in \mathbf{D}^\mathbf{N}} \sum_{j \in \mathbf{N}} \left| \sum_{i \in \mathbf{N}} A_{i,j} t_i \right| .$$

But this cannot be the case, since by lemma 27, one would be able to find $A \in \ell^1(\mathbf{N} \times \mathbf{N})$, $A \neq 0$, such that

$$\forall t \in \mathbf{D}^\mathbf{N} \quad \sum_{j \in \mathbf{N}} \left| \sum_{i \in \mathbf{N}} A_{i,j} t_i \right| \leq \frac{\|A\|_1}{2k} \leq \frac{1}{2} \sup_{s \in \mathbf{D}^\mathbf{N}} \sum_{j \in \mathbf{N}} \left| \sum_{i \in \mathbf{N}} A_{i,j} s_i \right|$$

and this is impossible since $\sup_{t \in \mathbf{D}^\mathbf{N}} \sum_{j \in \mathbf{N}} \left| \sum_{i \in \mathbf{N}} A_{i,j} t_i \right| > 0$ as $A \neq 0$.

So we can conclude with the following negative result.

Proposition 28 *There are Köthe spaces X and Y such that the topology of $X \multimap Y$ is strictly finer than the topology of uniform convergence on the equicontinuous subsets of E_X.*

And we are left with the problem of finding a functional characterization of the topology of $E_{X \multimap Y}$, but at least, a track is closed.

3 Exponentials and entire functions

If I is a set, we denote by $\mathcal{M}(I)$ the set of all finite multi-sets of elements of I. Let us first introduce some notations concerning finite multi-sets. If μ is an element of $\mathcal{M}(I)$, we define its *support* $|\mu|$ as the set of all $i \in I$ such that $\mu(i) \neq 0$. We define its *size* (or cardinality) as $\#\mu = \sum_{i \in I} \mu(i) \in \mathbf{N}$. We also define its *factorial* as $\mu! = \prod_{i \in I} \mu(i)!$. We extend to $\mathcal{M}(I)$ all the ordinary operations on natural numbers componentwise and denote by $[i_1, \ldots, i_n]$ the multi-set of the elements $i_1, \ldots, i_n$ of I, taking multiplicities into account. If $\mu, \nu \in \mathcal{M}(\mathbf{N})$ are such that $\nu \leq \mu$, we define the binomial coefficient

$$\binom{\mu}{\nu} = \frac{\mu!}{\nu!(\mu - \nu)!} = \prod_{i \in I} \binom{\mu(i)}{\nu(i)}$$

For $x \in \mathbf{K}^I$ and $\mu \in \mathcal{M}(I)$, we define $x^\mu \in \mathbf{K}$ as $x^\mu = \prod_{i \in I} x_i^{\mu(i)}$. Since the multi-set μ is finite, this product makes sense (we adopt the usual convention that $0^0 = 1$).

With these notations, the usual binomial equation immediately generalizes as follows: for $x, y \in \mathbf{K}^I$ and $\mu \in \mathcal{M}(I)$, one has $(x + y)^\mu = \sum_{\nu \leq \mu} \binom{\mu}{\nu} x^\nu y^{\mu - \nu}$.

Let now S be a commutative monoid (with additive notations for the operations). If $\gamma \in \mathcal{M}(S)$, we denote by $\Sigma(\gamma)$ the element of S given by $\Sigma(\gamma) = \sum_{s \in S} \gamma(s)s$.

We define next the multinomial coefficients for multi-sets. Let J be another index set. Let $\mu \in \mathcal{M}(I)$ and let $\alpha \in \mathcal{M}(I \times J)$. If the following property holds:

$$\forall i \in I \quad \sum_{j \in J} \alpha(i,j) = \mu(i)$$

then we define the multinomial coefficient

$$\begin{bmatrix}\mu \\ \alpha\end{bmatrix} = \frac{\mu!}{\alpha!} \in \mathbf{N}\,.$$

The binomial coefficient $\binom{\mu}{\nu}$ corresponds to the particular case $J = \{1,2\}$, $\alpha(i,1) = \nu(i)$ and $\alpha(i,2) = \mu(i) - \nu(i)$.

Let X be a Köthe space and let $x \in E_X$. We define then $x^! \in \mathbf{K}^{\mathcal{M}(|X|)}$ by $(x^!)_\mu = x^\mu$. The Köthe space $!X$ is given by

$$|!X| = \mathcal{M}(|X|) \quad \text{and} \quad E_{!X} = \{x^! \mid x \in E_X\}^{\perp\perp}\,.$$

Our definition of an entire map is just the natural extension to this setting of the usual definition of an entire map between finite dimensional spaces (analytic map with an infinite radius of convergence at 0).

Definition 29 Let X and Y be Köthe spaces. A function $h : E_X \to E_Y$ is *entire* if there exists a continuous linear function $f : E_{!X} \to E_Y$ such that $h(x) = f(x^!)$ for all $x \in E_X$. Such a linear function f (and its matrix) will be called a *power series* defining h. If $f : E_{!X} \to E_Y$ is a continuous linear function, we denote by $\mathcal{F}(f)$ the corresponding entire function from E_X to E_Y.

We shall first show that, when a function is entire, it is defined by a unique power series. This is done as usual, with the help of derivatives.

Let X be a Köthe space and let $f : E_X \to \mathbf{K}$ be an entire map, with defining power series given by its matrix $M \in E_{(!X)^\perp}$. This means that, for any $x \in E_X$, one has

$$f(x) = \sum_{\mu \in \mathcal{M}(|X|)} M_\mu x^\mu\,,$$

this sum converging absolutely. Let $s = (a_1, \dots, a_n)$ be a finite sequence of *pairwise distinct* elements of $|X|$. We define an injective continuous linear map $\eta_s : \mathbf{K}^n \to E_X$ by $\eta_s(t_1, \dots, t_n) = \sum_{i=1}^n t_i e_{a_i}$. Then $f \circ \eta_s$ is an entire function (in the usual sense) defined on $\mathbf{K}^n$ by the following absolutely converging sum, because the a_i's are pairwise distinct:

$$(f \circ \eta_s)(t_1, \dots, t_n) = \sum_{(k_1, \dots, k_n) \in \mathbf{N}^n} M_{k_1[a_1] + \cdots + k_n[a_n]} t_1^{k_1} \dots t_n^{k_n}\,.$$

Let $\mu \in \mathcal{M}(|X|)$, and let $s = (a_1, \dots, a_n)$ be an enumeration, without repetitions, of $|\mu|$ (so that n is the cardinality of $|\mu|$, not the cardinality of μ). Then M_μ appears as the coefficient of $t_1^{\mu(a_1)} \cdots t_n^{\mu(a_n)}$ in the development above of $(f \circ \eta_s)(t_1, \dots, t_n)$. Therefore

$$M_\mu = \frac{1}{\mu!} \frac{\partial^{\#\mu}(f \circ \eta_s)}{\partial t_1^{\mu(a_1)} \cdots \partial t_n^{\mu(a_n)}}(0, \dots, 0)$$

and this shows that an entire function from E_X to $\mathbf{K}$ has only one defining power series. This immediately extends to the case of the general entire functions $f : E_X \to E_Y$ (consider the entire functions $x \mapsto f(x)_b$ for $b \in |Y|$).

Proposition 30 *If $f : E_X \to E_Y$ is an entire function, there is exactly one linear continuous function $\widetilde{f} : E_{!X} \to E_Y$ such that $f(x) = \widetilde{f}(x^!)$ for each $x \in E_X$.*

In particular, there is a canonical linear isomorphism between $E_{(!X)^\perp}$ and $\mathcal{A}(X)$, the vector space of all entire mappings from E_X to $\mathbf{K}$. We denote by $\Theta_X : \mathcal{A}(X) \to E_{(!X)^\perp}$ this isomorphism (which maps an entire scalar-valued function to its defining power series). By proposition 30, this isomorphism is completely characterized by the following equation

$$\langle x^!, \Theta_X(h)\rangle = h(x)$$

which holds for all $x \in E_X$ and $h \in \mathcal{A}(X)$.

3.1 Functorial action of the exponentials

Given a continuous linear map $f : E_X \to E_Y$ of matrix $A = \mathsf{M}(f) \in E_{X \multimap Y}$, we want to define a continuous linear map $!f : E_{!X} \to E_{!Y}$ of matrix $!A \in E_{!X \multimap !Y}$. Rather than directly defining $!f$, we define its transpose $g = (!f)^\perp : E_{(!Y)^\perp} \to E_{(!X)^\perp}$. For this, we identify $E_{(!X)^\perp}$ with the space $\mathcal{A}(X)$ of all entire mappings from E_X to $\mathbf{K}$. Given $h \in \mathcal{A}(Y)$, we define $g(h) : E_X \to \mathbf{K}$ by

$$\forall x \in E_X \quad g(h)(x) = h(f(x)) \in \mathbf{K} \tag{2}$$

and we shall prove that $g(h)$ is entire. Let $\nu \in \mathcal{M}(|Y|)$ and consider the element $h \in \mathcal{A}(Y)$ which corresponds to the element $e_\nu \in E_{(!Y)^\perp}$, h is just the "monome" function $h(y) = y^\nu$. By condition (2), the matrix $!A$ must satisfy, for all $x \in E_X$

$$\sum_{\mu \in \mathcal{M}(|X|)} (!A)_{\mu,\nu} x^\mu = (f(x))^\nu = \prod_{b \in |Y|} \left(\sum_{a \in |X|} A_{a,b} x_a \right)^{\nu(b)} . \tag{3}$$

We shall need now the following lemma.

Lemma 31 *Let $(u_i)_{i \in I}$ be a family of elements of $\mathbf{K}$ such that the sum $\sum_{i \in I} u_i$ converges absolutely and let $k \in \mathbf{N}$. Then the sum*

$$\sum_{\substack{\gamma \in \mathcal{M}(I) \\ \#\gamma = k}} \begin{bmatrix} k \\ \gamma \end{bmatrix} u^\gamma$$

converges absolutely and its value is $\left(\sum_{i \in I} u_i\right)^k$.

Proof: The equality holds when u vanishes almost everywhere; this is standard. One concludes by continuity of the map $t \mapsto t^k$. ∎

So we can write

$$\begin{aligned} \left(\sum_{a \in |X|} A_{a,b} x_a \right)^{\nu(b)} &= \sum_{\substack{\rho \in \mathcal{M}(|X|) \\ \#\rho = \nu(b)}} \begin{bmatrix} \nu(b) \\ \rho \end{bmatrix} \prod_{a \in |X|} (A_{a,b} x_a)^{\rho(a)} \\ &= \sum_{\substack{\rho \in \mathcal{M}(|X|) \\ \#\rho = \nu(b)}} \begin{bmatrix} \nu(b) \\ \rho \end{bmatrix} \left(\prod_{a \in |X|} (A_{a,b})^{\rho(a)} \right) x^\rho \end{aligned}$$

Let $L(\nu)$ be the set of all multi-sets σ over the set $|X| \times |Y|$ such that $\sum_{a\in|X|} \sigma(a,b) = \nu(b)$ for each $b \in |Y|$ (these multi-sets σ are necessarily finite and by σ_b we denote the element of $\mathcal{M}(|X|)$ given by $\sigma_b(a) = \sigma(a,b)$).

The product $\prod_{b\in|Y|} \left(\sum_{a\in|X|} A_{a,b} x_a\right)^{\nu(b)}$ (which is actually a finite product over $|\nu|$) can now be written as the following absolutely converging sum:

$$\sum_{\sigma\in L(\nu)} \left(\prod_{b\in|\nu|} \begin{bmatrix}\nu(b)\\ \sigma_b\end{bmatrix} x^{\sigma_b}\right) \left(\prod_{(a,b)\in|X|\times|Y|} A_{a,b}^{\sigma_b(a)}\right) = \sum_{\sigma\in L(\nu)} \begin{bmatrix}\nu\\ \sigma\end{bmatrix} A^\sigma x^{\sum_{b\in|Y|}\sigma_b} .$$

So finally

$$\prod_{b\in|Y|} \left(\sum_{a\in|X|} A_{a,b} x_a\right)^{\nu(b)} = \sum_{\mu\in\mathcal{M}(|X|)} \left(\sum_{\sigma\in L(\mu,\nu)} \begin{bmatrix}\nu\\ \sigma\end{bmatrix} A^\sigma\right) x^\mu$$

where by $L(\mu,\nu)$ we denote the set of all multi-sets σ over $|X|\times|Y|$ such that $\sum_{a\in|X|}\sigma(a,b) = \nu(b)$ for each $b\in|Y|$ and $\sum_{b\in|Y|}\sigma(a,b) = \mu(a)$ for each $a\in|X|$; such multi-sets σ are $\mathbf{N}$-valued matrices with sums of columns prescribed by μ and sums of lines prescribed by ν. Observe that a necessary condition for $L(\mu,\nu)$ to be non empty is that $\#\mu = \#\nu$, and then $\#\sigma = \#\mu = \#\nu$ for each $\sigma\in L(\mu,\nu)$.

This gives us an explicit formula for the matrix $!A$:

$$(!A)_{\mu,\nu} = \sum_{\sigma\in L(\mu,\nu)} \begin{bmatrix}\nu\\ \sigma\end{bmatrix} A^\sigma . \tag{4}$$

Moreover, the above calculation shows that the sum $\sum_{\mu\in\mathcal{M}(|X|)} (!A)_{\mu,\nu} x^\mu$ converges absolutely, for each $x\in E_X$ and each $\nu\in\mathcal{M}(|Y|)$. Indeed, we can assume that A and x are positive, and we have obtained the sum $\sum_{\mu\in\mathcal{M}(|X|)} (!A)_{\mu,\nu} x^\mu$ as the finite product of converging sums of positive terms $\prod_{b\in|\nu|} \left(\sum_{a\in|X|} A_{a,b} x_a\right)^{\nu(b)}$. Assume still A and x to be positive, and let $R\in E_{(!Y)^\perp}$ be also positive, we show that the sum

$$\sum_{(\mu,\nu)\in\mathcal{M}(|X|)\times\mathcal{M}(|Y|)} (!A)_{\mu,\nu} x^\mu R_\nu$$

converges. Let $h\in\mathcal{A}(Y)$ be the entire function defined by R and let $f : E_X \to E_Y$ be the linear function defined by A. Then

$$\begin{aligned}
\infty > h(f(x)) &= \sum_{\nu\in\mathcal{M}(|Y|)} R_\nu (f(x))^\nu \\
&= \sum_{\nu\in\mathcal{M}(|Y|)} R_\nu \prod_{b\in|Y|} \left(\sum_{a\in|X|} A_{a,b} x_a\right)^{\nu(b)} \\
&= \sum_{\nu\in\mathcal{M}(|Y|)} R_\nu \left(\sum_{\mu\in\mathcal{M}(|X|)} (!A)_{\mu,\nu} x^\mu\right) \quad \text{by the calculation above} \\
&= \sum_{(\mu,\nu)\in\mathcal{M}(|X|)\times\mathcal{M}(|Y|)} (!A)_{\mu,\nu} x^\mu R_\nu
\end{aligned}$$

since we are dealing with sums of positive terms. Due to the following lemma, this implies that $!A\in E_{!X\multimap !Y}$.

Lemma 32 *Let X and Z be Köthe spaces and let $M \in \mathbf{K}^{|!X\multimap Z|}$. Then $M \in E_{!X\multimap Z}$ iff for all $x \in E_X$ and all $z' \in E_{Z^\perp}$, the sum $\sum_{\mu\in\mathcal{M}(|X|),c\in|Z|} |M_{\mu,c}x^\mu z'_c|$ converges.*

Proof: Assume that M satisfies the condition above. One observes first that for each $\mu \in |!X|$ and each $z' \in E_{Z^\perp}$ the sum $\sum_{c\in|Y|} |M_{\mu,c}z'_c|$ converges since it is majorized by the sum $\sum_{\nu\in\mathcal{M}(|X|),c\in|Z|} |M_{\nu,c}x^\nu z'_c|$ which converges by our assumption about M, where $x \in E_X$ is the characteristic map of $|\mu|$ (indeed $x^\mu = 1$). Then one concludes that $M \in E_{!X\multimap Z}$ like in the proof of lemma 20, using the fact that $E_{(!X)^\perp} = \{x^! \mid x \in E_X\}^\perp$. The converse implication is trivial. ∎

Therefore, if $A \in E_{X\multimap Y}$, $|!A| = !|A| \in E_{!X\multimap !Y}$ and so $!A \in E_{!X\multimap !Y}$.

Moreover, the calculation above shows that, for any $h \in \mathcal{A}(Y)$ and any linear and continuous $f : E_X \to E_Y$, one has $h(f(x)) = ((!A)^\perp R)x^!$ (product of matrices) for all $x \in E_X$, where $A \in E_{X\multimap Y}$ is the matrix of f and $R \in E_{(!Y)^\perp}$ is the matrix (the power series) of h.

To summarize:

Proposition 33 *Let $A \in E_{X\multimap Y}$. Then the matrix $!A$ defined by equation (4) belongs to $E_{!X\multimap !Y}$. Moreover, if $h \in \mathcal{A}(Y)$, then*

$$\widehat{(!A)^\perp}(\Theta_Y(h)) = \Theta_X(h \circ \widehat{A})\,. \tag{5}$$

If $f : E_X \to E_Y$ is linear and continuous, then $h \circ f : E_X \to \mathbf{K}$ is entire and

$$\widetilde{h \circ f} = \widetilde{h} \circ !f$$

where $!f$ is the linear continuous map from $E_{!X}$ to $E_{!Y}$ whose matrix is $!\mathsf{M}(f)$. Moreover, the operation $f \mapsto !f$ is functorial.

Lemma 34 *Let X and Y be Köthe spaces. One has*

$$E_{(!X\otimes !Y)^\perp} = \{x^! \otimes y^! \mid x \in E_X \text{ and } y \in E_Y\}^\perp$$

Proof: Since, for $x \in E_X$ and $y \in E_Y$, one has $x^! \otimes y^! \in E_{!X\otimes !Y}$, the inclusion $\subseteq$ holds. Let us prove the converse, so let $R \in \{x^! \otimes y^! \mid x \in E_X \text{ and } y \in E_Y\}^\perp$ and let $M \in E_{!X}$ and $N \in E_{!Y}$, we must show that the sum

$$T = \sum_{\mu\in|!X|,\nu\in|!Y|} |M_\mu N_\nu R_{\mu,\nu}|$$

converges. First, for any fixed $\nu \in |!Y|$, the sum $\sum_{\mu\in|!X|} |M_\mu R_{\mu,\nu}|$ converges as the family $(R_{\mu,\nu})_{\mu\in|!X|}$ belongs to $E_{(!X)^\perp}$. Indeed, let $x \in E_X$, we have to show that $S = \sum_{\mu\in|!X|} |x^\mu R_{\mu,\nu}|$ converges. But let $y : |Y| \to \mathbf{K}$ be the characteristic function of $|\nu|$, which belongs to E_Y since it vanishes almost everywhere. Then $y^\nu = 1$ and so the sum S is majorized by the sum $\sum_{\mu\in|!X|,\nu'\in|!Y|} \left|x^\mu y^{\nu'} R_{\mu,\nu'}\right|$ which converges by our hypothesis about R. So proving that T converges amounts to showing that the sum (whose terms are now well defined)

$$\sum_{\nu\in|!X|} |N_\nu| \sum_{\mu\in|!X|} |M_\mu R_{\mu,\nu}|$$

converges. For this, it is sufficient to show that the family $(\sum_{\mu\in|!X|} |M_\mu R_{\mu,\nu}|)_{\nu\in|!Y|}$ belongs to $E_{(!Y)^\perp}$. So let $y \in E_Y$, we must show that

$$T' = \sum_{\mu\in|!X|,\nu|!Y|} |M_\mu y^\nu R_{\mu,\nu}|$$

converges. The same considerations, on the other side, reduce our problem to showing that, for each $x \in E_X$ and each $y \in E_Y$, the sum $\sum_{\mu\in|!X|,\nu|!Y|} |x^\mu y^\nu R_{\mu,\nu}|$ converges, and this in turn results from our hypothesis about R. ∎

This exponential satisfies the following fundamental isomorphism.

Proposition 35 *Let X and Y be Köthe spaces. There is a canonical linear homeomorphism between $!(X \mathbin{\&} Y)$ and $!X \otimes !Y$.*

Proof: We assume $|X| \cap |Y| = \emptyset$ and we identify $|X \mathbin{\&} Y|$ with $|X| \cup |Y|$, just for simplifying the notations. The function $\varphi : |!X \otimes !Y| \to |!(X \mathbin{\&} Y)|$ which maps (λ, ρ) to $\lambda + \rho$ is a bijection, with inverse ψ. Then ψ^* is a linear homeomorphism $E_{(!X\otimes!Y)^\perp} \to E_{(!(X\&Y))^\perp}$ using lemma 14. This results from lemma 34 and from the observation that, for $x \in E_X$ and $y \in E_Y$, denoting by (x, y) the corresponding pair in $E_{X\&Y} \simeq E_X \times E_Y$, one has $(x, y)^\mu = x^\lambda y^\rho$, where $(\lambda, \rho) = \psi(\mu)$, that is

$$(x, y)^! = \psi^*(x^! \otimes y^!)\,.$$

∎

3.2 The co-monadic structure of the exponential

For each Köthe space X, we consider the two particular matrices (called respectively *dereliction* and *digging*): $\mathrm{d}^X \in \mathbf{K}^{|!X\multimap X|}$ and $\mathrm{p}^X \in \mathbf{K}^{!X\multimap !!X}$ given by

$$(\mathrm{d}^X)_{\mu,a} = \delta_{\mu,[a]} \quad \text{and} \quad (\mathrm{p}^X)_{\mu,M} = \delta_{\mu,\Sigma(M)}\,.$$

We recall that, if $M \in |!!X|$, then $\Sigma(M) \in |!X|$ is given by $\Sigma(M)(a) = \sum_{\nu\in|!X|} M(\nu)\nu(a)$.

Let $x \in E_X$ and let $x' \in E_{X^\perp}$, we have

$$\sum_{\mu\in|!X|,a\in|X|} \left|x^\mu x'_a (\mathrm{d}^X)_{\mu,a}\right| = \sum_{a\in|X|} \left|x_a x'_a\right|$$

and hence $\mathrm{d}^X \in E_{!X\multimap X}$.

Remark: This matrix d^X allows us to see any linear continuous map as an entire map. Indeed, let $f : E_X \to E_Y$ be linear and continuous. Then $f \circ \widehat{\mathrm{d}^X} : E_{!X} \to E_Y$ is linear and continuous, and so defines an entire map $h : E_X \to E_Y$ by $h(x) = f(\widehat{\mathrm{d}^X}(x^!))$, that is $h(x) = f(x)$. We shall say that an entire map $h : E_X \to E_Y$ is *linear* when $h = f \circ \widehat{\mathrm{d}^X}$ for some linear and continuous f, and then one easily sees that $f = \widetilde{h}$. One checks also easily that an entire map h is linear iff it is linear in the usual sense, that is $h(\lambda x) = \lambda h(x)$ and $h(x + y) = h(x) + h(y)$ for $\lambda \in \mathbf{K}$ and $x, y \in E_X$.

Now, let $x \in E_X$ and let $R \in E_{(!!X)^\perp}$, we have

$$\sum_{\mu\in|!X|,M\in|!!X|} \left|x^\mu R_M (\mathrm{p}^X)_{\mu,M}\right| = \sum_{M\in|!!X|} \left|x^{\Sigma(M)} R_M\right|\,.$$

But $(x^!)^! \in E_{!!X}$ and one has $((x^!)^!)_M = x^{\Sigma(M)}$ for each $M \in |!!X|$ and so the sum above converges, by our hypothesis about R. Therefore, $\mathrm{p}^X \in E_{!X\multimap !!X}$.

Checking that d^X is natural in X is a straightforward verification. We check that p^X is natural in X. Let $A \in E_{X\multimap Y}$, we must show that the following equation of matrices holds:

$$\mathrm{p}^Y(!A) = (!!A)\mathrm{p}^X\,.$$

This amounts to showing that, for each $\mu \in |!X|$ and each $N \in |!!Y|$, one has

$$(!A)_{\mu,\Sigma(N)} = \sum_{\substack{M\in|!!X|\\ \Sigma(M)=\mu}} (!!A)_{M,N}\,.$$

This latter sum is a finite sum as indeed $(!!A)_{M,N} \neq 0 \Rightarrow \#M = \#N$ and there are only finitely many multisets $M \in |!!X|$ such that $\#M = \#N$ and $\Sigma(M) = \mu$. Let $t \in E_{!X}$. Equation (3) gives in that case

$$\sum_{M\in|!!X|} (!!A)_{M,N} t^M = \prod_{\nu\in|!Y|} \left(\sum_{\mu\in|!X|} (!A)_{\mu,\nu} t_\mu\right)^{N(\nu)}$$

Let $x \in E_X$ and replace in this equation t by $x^!$, we obtain

$$\sum_{\rho\in|!X|} \left(\sum_{\substack{M\in|!!X|\\ \Sigma(M)=\rho}} (!!A)_{M,N}\right) x^\rho = \prod_{\nu\in|!Y|} \left(\sum_{\mu\in|!X|} (!A)_{\mu,\nu} x^\mu\right)^{N(\nu)} \tag{6}$$

For each $\nu \in |!Y|$, we have $\sum_{\mu\in|!X|}(!A)_{\mu,\nu}x^\mu = (Ax)^\nu$ by equation (3), and hence the right-hand side of equation (6) is equal to $(Ax)^{\Sigma(N)}$, and so, applying once more equation (3), we arrive to

$$\sum_{\rho\in|!X|} \left(\sum_{\substack{M\in|!!X|\\ \Sigma(M)=\rho}} (!!A)_{M,N}\right) x^\rho = \sum_{\rho\in|!X|} (!A)_{\rho,\Sigma(N)} x^\rho$$

and we conclude by proposition 30.

To prove that, with the natural transformations d^X and p^X, the functor $X \mapsto !X$ is a comonad, one has to check the commutations of three diagrams corresponding to the following equations

$$(!\mathrm{d}^X)\mathrm{p}^X = I_{!X} \quad \text{and} \quad \mathrm{d}^{!X}\mathrm{p}^X = I_{!X}$$

and

$$(!\mathrm{p}^X)\mathrm{p}^X = \mathrm{p}^{!X}\mathrm{p}^X$$

We check the third equation; the first one is similar (but simpler) and the second one is trivial.

Let $M \in |!!X|$ and $\mathcal{S} \in |!!!X|$. For $P \in \mathcal{M}(|!X| \times |!!X|)$, one has $(\mathrm{p}^X)^P \neq 0$ iff $|P| \subseteq \{(\rho, R) \in |!X| \times |!!X| \mid \Sigma(R) = \rho\} = \mathcal{H}$, by definition of p^X. Now let $P \in \mathcal{M}(|!X| \times |!!X|)$ be such that $|P| \subseteq \mathcal{H}$. Then $P \in L(M, \mathcal{S})$ (with the notations introduced for equation (4)) holds iff, for all $R \in |!!X|$, $\mathcal{S}(R) = P(\Sigma(R), R)$ and, for all $\rho \in |!X|$,

$$M(\rho) = \sum_{R\in|!!X|} P(\rho, R) = \sum_{\Sigma(R)=\rho} \mathcal{S}(R)$$

If M and $\mathcal{S}$ satisfy that $M(\rho) = \sum_{\Sigma(R)=\rho} \mathcal{S}(R)$ for each $\rho \in |!X|$, then $L(M, \mathcal{S})$ has exactly one element P such that $|P| \subseteq \mathcal{H}$, and P is given by $P(\rho, R) = \mathcal{S}(R)\delta_{\rho,\Sigma(R)}$. Moreover, if P and $\mathcal{S}$ satisfy $\mathcal{S}(R) = P(\Sigma(R), R)$ for all $R \in |!!X|$ and if $|P| \subseteq \mathcal{H}$, then $\left[\begin{smallmatrix}\mathcal{S}\\ P\end{smallmatrix}\right] = 1$. Therefore formula (4) gives in the present situation

$$(!\mathrm{p}^X)_{M,\mathcal{S}} = \begin{cases} 1 & \text{if } \forall\rho \in |!X|\ M(\rho) = \sum_{\Sigma(R)=\rho} \mathcal{S}(R) \\ 0 & \text{otherwise} \end{cases}$$

Given $\mu \in |!X|$ and $\mathcal{S} \in |!!!X|$, we have therefore $((!\mathrm{p}^X)\mathrm{p}^X)_{\mu,\mathcal{S}} = \sum_{\Sigma(M)=\mu}(!\mathrm{p}^X)_{M,\mathcal{S}} = \delta_{\mu,T(\mathcal{S})}$ where $T(\mathcal{S}) \in |!X|$ is given by

$$
\begin{aligned}
T(\mathcal{S})(a) &= \sum_{\rho\in|!X|} \rho(a) \sum_{\Sigma(R)=\rho} \mathcal{S}(R) \\
&= \sum_{R\in|!!X|} (\Sigma(R))(a)\mathcal{S}(R) \\
&= \sum_{R\in|!!X|} \left(\sum_{\rho\in|!X|} \rho(a)R(\rho) \right) \mathcal{S}(R) \\
&= \sum_{\rho\in|!X|, R\in|!!X|} \rho(a)R(\rho)\mathcal{S}(R) \\
&= \Sigma(\Sigma(\mathcal{S}))(a)\,,
\end{aligned}
$$

but it is clear on the other hand that $((\mathrm{p}^{!X})\mathrm{p}^X)_{\mu,\mathcal{S}} = \delta_{\mu,\Sigma(\Sigma(\mathcal{S}))}$ and this concludes the proof.

3.3 The co-algebraic structure of the exponential

The diagonal continuous linear map $D_X : X \to X \,\&\, X$ maps $x \in E_X$ to $(x,x) \in E_X \times E_X \simeq E_{X\&X}$, and its matrix is given by $\mathsf{M}(D_X)_{a,(i,b)} = \delta_{a,b}$. Then $!(D_X)$ induces, through the isomorphism of proposition 35, a linear continuous map $\Delta_X : !X \to !X \otimes !X$ (corresponding to *contraction* in logic). The matrix of this operator is easily seen to be given by

$$\mathsf{M}(\Delta_X)_{\mu,(\lambda,\rho)} = \delta_{\mu,\lambda+\rho}$$

for $\mu, \lambda, \rho \in |!X|$. This turns $!X$ into a co-algebra, with neutral element e_0 (where 0 denotes the empty multi-set), and Δ_X is associative and commutative, because D_X is, trivially. Categorically, the neutral element must be considered as the linear continuous map $\mathrm{w}_X : E_{!X} \to E_1 = \mathbf{K}$ given by $\mathrm{w}_X(x) = x_0$ (corresponding to *weakening* in logic). In more algebraic notations, the action of Δ_X on an element t of $E_{!X}$ can be written

$$\Delta_X(t) = \sum_{\lambda,\rho\in|!X|} t_{\lambda+\rho}(e_\lambda \otimes e_\rho)\,.$$

In particular $\Delta_X(x^!) = x^! \otimes x^!$ for $x \in E_X$.

Until now, we have shown that the category of Köthe spaces and linear continuous maps (equipped with the tensor product and the exponential functor we have presented) is a model of first order propositional linear logic, and more precisely, a *new-Seely catgory* in the sense of [Bie95].

3.4 The Hopf algebra structure of the exponential

We apply proposition 35, and the fact that finite sums and cartesian products of Köthe spaces coincide for endowing the space $!X$ with an *algebraic* structure. Indeed, the co-diagonal morphism $X \oplus X \to X$ induces a linear continuous map $\mathrm{a}_X : X \,\&\, X \to X$ given by $\mathrm{a}_X(x,y) = x+y$. Its matrix is given by $\mathsf{M}(\mathrm{a}_X)_{(i,a),b} = \delta_{a,b}$. Then $!(\mathrm{a}_X)$ induces, through the isomorphism of proposition 35, a linear continuous map $\mathrm{m}_X : !X \otimes !X \to !X$. An easy computation, using formula (4), shows that, for $\lambda, \rho, \mu \in |!X|$,

$$\mathsf{M}(\mathrm{m}_X)_{(\lambda,\rho),\mu} = \binom{\mu}{\lambda} \delta_{\lambda+\rho,\mu}\,. \tag{7}$$

The neutral element of this operation is $e_0 \in E_{!X}$. The associativity of this operation immediately results from its definition, and from the associativity of addition. Commutativity of m_X is established in the same way. If we consider m_X as a bilinear map $E_{!X} \times E_{!X} \to E_{!X}$, its action on $(s,t) \in E_{!X} \times E_{!X}$ can be written

$$\mathrm{m}_X(s,t) = \sum_{\lambda,\rho \in |!X|} \binom{\lambda+\rho}{\lambda} s_\lambda t_\rho e_{\lambda+\rho} = \sum_{\mu \in |!X|} \left(\sum_{\lambda \leq \mu} \binom{\mu}{\lambda} s_\lambda t_{\mu-\lambda} \right) e_\mu \,.$$

In particular, $\mathrm{m}_X(x^!, y^!) = (x+y)^!$, for $x, y \in E_X$.

Keeping implicit the linear homeomorphism between $!X \otimes !X$ and $!(X \mathbin{\&} X)$ and the linear isomorphism between $E_{(!X)^\perp}$ and $\mathcal{A}(X)$, the dual of m_X maps an element h of $\mathcal{A}(X)$ to the element k of $\mathcal{A}(X \mathbin{\&} X)$ defined by $k(x,y) = h(x+y)$.

The last structure map of a Hopf algebra is the antipode S_X. Let $\mathrm{n}_X : E_X \to E_X$ be the continuous linear map defined by $\mathrm{n}_X(x) = -x$, its matrix is given by $\mathsf{M}(\mathrm{n}_X)_{a,b} = -\delta_{a,b}$. The antipode is defined as $\mathrm{S}_X = !\mathrm{n}_X$, and it is a linear continuous map from $!X$ to itself. Its matrix is easily seen (again, using formula (4)) to be given by

$$\mathsf{M}(\mathrm{S}_X)_{\mu,\nu} = (-1)^{\#\mu} \delta_{\mu,\nu} \,.$$

In other words, $\mathrm{S}_X(t) = \sum_{\mu \in |!X|} (-1)^{\#\mu} t_\mu e_\mu$ for all $t \in E_{!X}$.

Checking that $(!X, \mathrm{m}_X, \Delta_X, \mathrm{S}_X)$ is a Hopf algebra consists first in checking that $(!X, \mathrm{m}_X, \Delta_X)$ is a bi-algebra. This is done using straightforwardly the definitions of m_X, Δ_X and the functoriality of !. The last diagram to check expresses that S_X is an antipode, and this is also trivial.

4 A cartesian closed category of entire mappings

The standard co-Kleisli construction applied to the co-monad $X \mapsto !X$ gives rise to a cartesian closed category $\mathcal{K}^!_{\mathbf{K}}$ which is defined as follows:

- an object in this category is a Köthe space;
- a morphism from X to Y in this category is a continuous linear map from $E_{!X}$ to E_Y;
- the identity from X to X in $\mathcal{K}^!_{\mathbf{K}}$ is $\widehat{\mathrm{d}^X} \in \mathcal{L}(E_{!X}, E_X)$;
- if $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathcal{K}^!_{\mathbf{K}}$, their composite $g \circ_{\mathcal{A}} f$ in $\mathcal{K}^!_{\mathbf{K}}$ is given by $g \circ_{\mathcal{A}} f = g \circ !f \circ \widehat{\mathrm{p}^X}$ where composition is taken in $\mathcal{K}_{\mathbf{K}}$:

$$E_{!X} \xrightarrow{\widehat{\mathrm{p}^X}} E_{!!X} \xrightarrow{!f} E_{!Y} \xrightarrow{g} E_Z \,.$$

So there is a bijective correspondence between the morphisms $f : X \to Y$ in $\mathcal{K}^!_{\mathbf{K}}$ and the entire functions from E_X to E_Y. We show first that the definition above of identity and composition is compatible with this correspondence. Remember that, if $f : E_{!X} \to E_Y$ is linear and continuous, we denote by $\mathcal{F}(f) : E_X \to E_Y$ the corresponding entire function (see definition 29).

As to the identity, given $x \in E_X$, we have

$$\begin{aligned}
\mathcal{F}(\widehat{\mathrm{d}^X})(x) &= \widehat{\mathrm{d}^X}(x^!) \\
&= \sum_{a\in|X|}\left(\sum_{\mu\in|!X|}(\mathrm{d}^X)_{\mu,a}(x^!)_\mu\right)e_a \\
&= \sum_{a\in|X|}(x^!)_{[a]}e_a \\
&= x
\end{aligned}$$

since $(x^!)_{[a]} = x_a$.

As to composition, we must show that (with the notations above), $\mathcal{F}(g \circ_{\mathcal{A}} f) = \mathcal{F}(g) \circ \mathcal{F}(f)$. This results immediately from the following

Lemma 36 *If $x \in E_X$, then $(!f)(\widehat{\mathrm{p}^X}(x^!)) = f(x^!)^!$.*

Proof: Denoting by A the matrix of f (so that $A \in E_{!X\multimap Y}$), we have

$$\begin{aligned}
(!f)(\widehat{\mathrm{p}^X}(x^!)) &= \sum_{M\in|!!X|} x^{\Sigma(M)}(!f)(e_M) \\
&= \sum_{M\in|!!X|} x^{\Sigma(M)}\left(\sum_{\nu\in|!Y|}(!A)_{M,\nu}e_\nu\right) \\
&= \sum_{\nu\in|!Y|}\left(\sum_{M\in|!!X|}(!A)_{M,\nu}x^{\Sigma(M)}\right)e_\nu \\
&= \sum_{\nu\in|!Y|}\left(\sum_{M\in|!!X|}(!A)_{M,\nu}(x^!)^M\right)e_\nu \\
&= \sum_{\nu\in|!Y|} f(x^!)^\nu e_\nu \quad \text{by equation (3)} \\
&= f(x^!)^!
\end{aligned}$$

and we are done. ∎

The cartesian product of X and Y in $\mathcal{K}^!_{\mathbf{K}}$ is $X \mathbin{\&} Y$. If $f : E_{X\&Y} \to E_Z$ is entire, its "transpose" (curryfication) is $f^c : E_X \to E_{!Y\multimap Z}$ where $f^c(x) = \mathsf{M}(\widetilde{g})$, where $g : E_Y \to E_Z$ is the entire function defined by $g(y) = f(x,y)$. The function f^c itself is entire, and we have $\mathsf{M}(\widetilde{f^c})_{\mu,(\nu,c)} = \mathsf{M}(\widetilde{f})_{\mu+\nu,c}$ where $\mu+\nu$ is the multi-set on $|X|+|Y|$ obtained by juxtaposing $\mu \in |!X|$ and $\nu \in |!Y|$. There is also an entire evaluation map $\mathrm{ev} : E_{(!X\multimap Y)\&X} \to E_Y$ given by $\mathrm{ev}(f,x) = f(x)$; its power series (identifying the canonically isomorphic spaces $!((!X \multimap Y) \mathbin{\&} X)$ and $!(!X \multimap Y) \otimes !X$) is given by the matrix

$$\mathsf{M}(\mathrm{ev})_{\varphi,\mu,b} = \delta_{\varphi,[(\mu,b)]}\,.$$

4.1 Computing derivatives

The space $!X$ possesses another important structure which is a map $\partial_X : E_X \to E_{!X}$ given by

$$\partial_X(x)_\mu = \begin{cases} x_a & \text{if } \mu = [a] \text{ for some } a \in |X| \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $\partial_X \in \mathcal{L}(E_X, E_{!X})$, with matrix $\mathsf{M}(\partial_X)_{a,\mu} = \delta_{[a],\mu}$. This map is natural in X and one has

$$\widehat{\mathrm{d}^X} \circ \partial_X = \mathrm{Id}_X\,. \tag{8}$$

Given an entire function $f : E_X \to E_Y$, $\widetilde{f} \circ \partial_X$ is the linear continuous map from E_X to E_Y whose matrix is given by $\mathsf{M}(\widetilde{f} \circ \partial_X)_{a,b} = \widetilde{f}_{[a],b}$, that is, $\widetilde{f} \circ \partial_X$ is the derivative of f at 0. More generally, one can define a map ∂^1_X as the composite of the two maps

$$!X \otimes X \xrightarrow{!X \otimes \partial_X} !X \otimes !X \xrightarrow{\mathrm{m}_X} !X$$

and an easy computation using equation (7) shows that, for $\mu, \nu \in |!X|$ and $a \in |X|$, one has $(\partial^1_X)_{\mu,a,\nu} = \nu(a)\delta_{\mu+[a],\nu}$. Therefore, if $f : E_X \to E_Y$ is entire, the composite $\widetilde{f} \circ \partial^1_X$ is a continuous linear map $!X \otimes X \to Y$ whose transpose $Df : !X \to (X \multimap Y)$ (using the monoidal closeness of $\mathcal{K}_{\mathbf{K}}$) is the derivative of f (it is an entire map from E_X to $E_{X\multimap Y}$), in the sense of the formal derivatives of power series. In particular, for all $y' \in E_{Y^\perp}$ and $u \in E_X$ the map $t \mapsto \langle f(x+tu), y'\rangle$ from $\mathbf{K}$ to $\mathbf{K}$ is entire (as the composite of three entire maps) and one checks easily that

$$\langle Df(x) \cdot u, y'\rangle = \frac{d\langle f(x+tu), y'\rangle}{dt}(0)$$

where, if $A \in E_{X\multimap Y}$ and $u \in E_X$, we recall that we denote by $A \cdot u$ the application of the matrix A to the vector u. This property completely characterizes the derivative Df of f.

Observe also that when $f : E_X \to E_Y$ is linear, then we have

$$\forall x \in E_X \quad Df(x) = \mathsf{M}(f)\,. \tag{9}$$

Of course, we can compute n-th derivatives for all $n \in \mathbf{N}$, and this is done by precomposing $\widetilde{f}$ with the linear continuous map $\partial^n_X : !X \otimes X^{\otimes n} \to !X$ obtained by iterating ∂^1_X in the obvious way, setting $\partial^{n+1}_X = \partial^n_X \circ (\partial^1_X \otimes X^{\otimes n})$:

$$!X \otimes X \otimes X^{\otimes n} \xrightarrow{\partial^1_X \otimes X^{\otimes n}} !X \otimes X^{\otimes n} \xrightarrow{\partial^n_X} !X\,.$$

Again, an easy induction shows that

$$(\partial^n_X)_{\mu,a_1,\dots,a_n,\nu} = \frac{\nu!}{\mu!}\delta_{\mu+[a_1,\dots,a_n],\nu}\,. \tag{10}$$

If $f : E_X \to E_Y$ is entire, its n-th derivative $D^n f$ is the transpose of the linear continuous map $\widetilde{f} \circ \partial^n_X : E_{!X\otimes X^{\otimes n}} \to E_Y$, considered as an entire map from E_X to $E_{X^{\otimes n}\multimap Y}$. Of course, for each $x \in E_X$, the corresponding n-linear map $D^n f(x) : E^n_X \to E_Y$ is symmetrical.

Taylor formula. One proves then the Taylor formula: for each $x \in E_X$,

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(0)(x^{\otimes n})\,. \tag{11}$$

Let $A \in E_{!X\multimap Y}$ be the power series defining f, we have

$$\begin{aligned} f(x) &= \sum_{\mu\in|!X|,\, b\in|Y|} A_{\mu,b} x^\mu e_b \\ &= \sum_{n=0}^{\infty} \left(\sum_{\#\mu=n,\, b} A_{\mu,b} x^\mu e_b \right) \end{aligned}$$

but for an element μ of $|!X|$ such that $\#\mu = n$, there are exactly $n!/\mu!$ tuples $(a_1, \dots, a_n)$ such that $[a_1, \dots, a_n] = \mu$. Moreover, for such a tuple, we have $x^\mu = (x^{\otimes n})_{(a_1,\dots,a_n)}$. Applying equation (10) and the definition above of $D^n f$, we see that (with μ and $(a_1, \dots, a_n)$ as before, and with $b \in |Y|$):

$$(D^n f(0))_{(a_1,\dots,a_n),b} = \mu!\, A_{\mu,b}\,,$$

so

$$f(x) = \sum_{n=0}^{\infty} \sum_{\substack{(a_1,\dots,a_n)\in|X^{\otimes n}|\\ b\in|Y|}} \frac{[a_1, \dots, a_n]!}{n!} \frac{1}{[a_1, \dots, a_n]!} (D^n f(0))_{(a_1,\dots,a_n),b} (x^{\otimes n})_{(a_1,\dots,a_n)}$$

and we conclude.

The Taylor series (11) converges absolutely in E_Y for each x (this means that for each $y' \in E_{Y^\perp}$, the series of positive terms $\sum_{n=0}^{\infty} N_{y'}(\frac{1}{n!} D^n f(0)(x^{\otimes n}))$ converges). Indeed,

$$\frac{1}{n!} D^n f(0)(x^{\otimes n}) = \sum_{\#\mu=n,\, b} A_{\mu,b} x^\mu e_b$$

so that $N_{y'}(\frac{1}{n!} D^n f(0)(x^{\otimes n})) \le \sum_{\#\mu=n,\, b} |y'_b A_{\mu,b} x^\mu|$ and we conclude since we know that $\sum_{\mu,\, b} |y'_b A_{\mu,b} x^\mu| < \infty$ because f is entire.

One can say better, namely that the the series $\sum_n f_n$ converges absolutely in $E_{!X \multimap Y}$, where f_n is the entire (indeed "polynomial") map defined by $f_n(x) = \frac{1}{n!} D^n f(0)(x^{\otimes n})$.

Partial derivatives. When $f : E_X \to E_Z$ is entire, we also denote by

$$\frac{df}{dx} : E_X \to E_{X \multimap Z} \quad \text{and} \quad \frac{d^n f}{dx^n} : E_X \to E_{X^{\otimes n} \multimap Z}$$

its derivatives.

Consider now an entire map of two variables $f : E_{X \& Y} \to E_Z$. Using the cartesian closeness of the category of Köthe spaces and entire mappings, we can transpose this map into an entire map $f_1 : E_X \to E_{!Y \multimap Z}$ which in turn can be derived, giving rise to the entire map

$$\frac{df_1}{dx} : E_X \to E_{X \otimes !Y \multimap Z}$$

(we have used the monoidal closeness of $\mathcal{K}_{\mathbf{K}}$), which, when considered as a linear continuous map $\widetilde{\frac{df_1}{dx}} : E_{!X} \to E_{X \otimes !Y \multimap Z}$, can be transposed into a linear continuous map (up to the symmetry of $\otimes$) $E_{!X \otimes !Y} \to E_{X \multimap Z}$ giving rise to an entire mapping

$$\frac{\partial f}{\partial x} : E_{X \& Y} \to E_{X \multimap Z}\,,$$

the partial derivative of f with respect to the first parameter. One defines of course similarly

$$\frac{\partial f}{\partial y} : E_{X \& Y} \to E_{Y \multimap Z}\,.$$

Then the derivative

$$\frac{df}{d(x,y)} : E_{X \& Y} \to E_{X \& Y \multimap Z}$$

can be expressed as follows, using these partial derivatives:

$$\frac{df}{d(x,y)}(x,y) \cdot (u,v) = \frac{\partial f}{\partial x}(x,y) \cdot u + \frac{\partial f}{\partial y}(x,y) \cdot v \in E_Z \tag{12}$$

for $x, u \in E_X$ and $y, v \in E_Y$. Of course, this can be generalized in the obvious way to entire functions depending on a finite number of parameters.

Chain rule. Consider now two entire mappings $f : E_X \to E_Y$ and $g : E_Y \to E_Z$. Then the usual chain rule holds, that is, the derivative of $g \circ f$ satisfies

$$\frac{d(g \circ f)}{dx}(x) \cdot u = \frac{dg}{dy}(f(x)) \cdot \left(\frac{df}{dx}(x) \cdot u\right). \tag{13}$$

Contraction and evaluation. Let $f : E_{X \& X} \to E_Z$ be an entire map of two parameters in the same space E_X (the first generic parameter will be denoted by "x", the second by "y"). The map $g : E_X \to E_Z$ defined by $g(x) = f(x,x)$ is entire: $\widetilde{g}$ is obtained by composing $\widetilde{f}$ with $!D_X : E_{!X} \to E_{!(X \& X)}$, a linear continuous map, i.e. by composing f with the diagonal $X \to (X \,\&\, X)$ in $\mathcal{K}^!_{\mathbf{K}}$. Using formulae (12) and (13), we obtain easily

$$\frac{dg}{dx}(x) \cdot u = \frac{\partial f}{\partial x}(x,x) \cdot u + \frac{\partial f}{\partial y}(x,x) \cdot u \tag{14}$$

which will be useful for derivating proofs containing contractions.

Using the matricial characterization above of the evaluation map $\mathrm{ev} : E_{(!X \multimap Y) \& X} \to E_Y$, one shows easily that the two partial derivatives of this entire map are given by

$$\frac{\partial \mathrm{ev}}{\partial f}(f,x) \cdot h = h(x) \tag{15}$$

and

$$\frac{\partial \mathrm{ev}}{\partial x}(f,x) \cdot u = \frac{df}{dx}(x) \cdot u \tag{16}$$

for $f, h \in E_{!X \multimap Y}$ and $x, u \in E_X$ (in the latter formula, we identify an element of $E_{!X \multimap Y}$ with an entire map from E_X to E_Y). The first partial derivative does not depend on f; this corresponds to the fact that evaluation is linear in its first argument. Observe also that the second derivative is linear in its first argument as well.

Remark: All these easy properties give a quite natural semantical foundation to an extension of proof systems and typed λ-calculus by differential constructions. With L. Regnier, we have been able in that way to define a *differential lambda-calculus* where lambda-terms can be formally differentiated, see [ER01].

5 Towards an intrinsic theory

All the spaces introduced so far were equipped with *explicit bases* (the webs) which, from the viewpoint of classical functional analysis, seems to be a major drawback of our approach. The goal of this section is to show that, at least in principle, these bases are not essential.

We shall say that a topological vector space E is an *intrinsic Köthe space* if there is a Köthe space X and a linear homeomorphism $\varphi : E \to E_X$. Such a linear homeomorphism φ will be called a *chart* of E. Such a chart is by no mean part of the structure of E; only the existence of a chart turns a topological vector space E into an intrinsic Köthe space, and the existence of a chart is a property of the topology of E. The goal of this section is to show that the constructions defined so far can be performed on intrinsic Köthe spaces, and not only on Köthe spaces. This means in some sense that they are "basis independent".

Let E be a topological vector space and let X be a Köthe space. Let $\varphi : E \to E_X$ be a linear homeomorphism. Through this isomorphism, E inherits a bornology $\mathcal{E}_\varphi$ from the equicontinuous bornology $\mathcal{E}_X$ of E_X:

$$\mathcal{E}_\varphi = \{\varphi^{-1}(G) \mid G \in \mathcal{E}_X\} = \{H \subseteq E \mid \varphi(H) \in \mathcal{E}_X\}\,.$$

Consider now some other linear homeomorphism $\psi : E \to E_Y$ to a Köthe space. We claim that $\mathcal{E}_\psi \subseteq \mathcal{E}_\varphi$, and so, by symmetry, $\mathcal{E}_\psi = \mathcal{E}_\varphi$. Let $H \subseteq E$ be such that $H \in \mathcal{E}_\psi$, that is, $\psi(H) \in \mathcal{E}_Y$. Let θ be the linear homeomorphism $\varphi \circ \psi^{-1} : E_Y \to E_X$. Then $\varphi(H) = \theta(\psi(H))$ and we conclude by proposition 18.

So an intrinsic Köthe space E possesses a canonical equicontinuous "bornology" (class of bounded sets, see [Jar81, Sch73]), that we can denote by $\mathcal{E}_E$, and which is equal to $\mathcal{E}_\varphi$ where φ is any chart of E. Moreover, E is clearly locally convex, Hausdorff, separable and complete and also, E possesses an absolute basis (it is shown in [Jar81], chapter 14, that these properties are equivalent to saying that E is linearly homeomorphic to a Köthe sequence space $\Lambda(P)$, but they are not sufficient for implying that E is an intrinsic Köthe space in our sense).

Remark: One can rephrase the definition of an intrinsic Köthe space as follows: a topological vector space E is an intrinsic Köthe space if it is a complete Hausdorff lcs which admits an absolute basis (x_n) such that, for any sequence r_n of non-negative real numbers, if the series $\sum_n |\langle u_n, x\rangle|\, r_n$ converges for each $x \in E$, then the semi-norm $x \mapsto \sum_n |\langle u_n, x\rangle|\, r_n$ is continuous, where $u_n : E \to \mathbf{K}$ is the n-th coefficient linear form associated with (x_n).

5.1 Topological dual

Using this bornology, we can define the topological dual E' of E as a topological vector space: E' is the linear space of all continuous linear forms on E, and a subset U' of E' is a neighborhood of 0 iff there exists $G \in \mathcal{E}_E$ such that, for any $f \in E'$, if $|f(x)| < 1$ for each $x \in G$, then $f \in U'$ (that is, the topology of E' is the topology of uniform convergence on the elements of $\mathcal{E}_E$). If $u \in E$ and $u' \in E'$, then we denote by $\langle u, u'\rangle$ the application of u' to u.

Lemma 37 *The function $\varphi' : E' \to E_{X^\perp}$ which maps each $u' \in E'$ to $\mathsf{M}(u' \circ \varphi^{-1})$ is a linear homeomorphism from E' to $E_{X^\perp}$. If $x \in E_X$, one has*

$$\langle x, \varphi'(u')\rangle = \langle \varphi^{-1}(x), u'\rangle$$

and if $x' \in E_{X^\perp}$, one has

$$\langle u, {\varphi'}^{-1}(x')\rangle = \langle \varphi(u), x'\rangle\,.$$

This results from proposition 17. One checks easily that if $\varphi : E \to E_X$ is a linear homeomorphism from E to some Köthe space, then $\langle u, u'\rangle = \langle \varphi(u), \varphi'(u')\rangle$.

So E' is an intrinsic Köthe space and possesses therefore a canonical bornology $\mathcal{E}_{E'}$, and the reader can check that a subset G' of E' belongs to this bornology iff it is an equicontinuous set of linear forms. Moreover, this operation $E \mapsto E'$ is functorial; let indeed $f : E \to F$ be linear and continuous. Its usual algebraic transpose ${}^{\mathrm{t}}f : F' \to E'$, which maps $v' \in F'$ to $v' \circ f \in E'$, is linear. Its continuity results (for instance) from the characterization above of the dual topology, and from the fact that continuous linear maps preserve equicontinuous sets (see proposition 18).

Last, the canonical linear function $\eta : E \to E''$ which maps $u \in E$ to the linear form $\eta(u)$ on E' given by $\eta(u)(u') = u'(u)$ is easily seen to make the following diagram commutative

$$\begin{tikzcd}
E \arrow[dd, "\eta"'] \arrow[dr, "\varphi"] & \\
& E_X = E_{X^{\perp\perp}} \\
E'' \arrow[ur, "\varphi''"] &
\end{tikzcd}$$

so that $\eta = {\varphi''}^{-1} \circ \varphi$ is a linear homeomorphism.

The cartesian product of two intrinsic Köthe spaces is an intrinsic Köthe space, endowed with the product topology, which is also the direct sum of the two spaces.

5.2 Linear function space

Now we study the space $\mathcal{L}(E,F)$ of continuous linear maps from E to F, two intrinsic Köthe spaces[7]. This is clearly a vector space, and we must endow it with a topology. Let $\varphi : E \to E_X$ and $\psi : F \to E_Y$ be linear homeomorphisms to some Köthe spaces X and Y. We shall say that a subset $\mathcal{U}$ of $\mathcal{L}(E,F)$ is $\varphi\psi$-open if the set $\mathcal{U}_{\varphi,\psi} = \{\mathsf{M}(\psi \circ f \circ \varphi^{-1}) \mid f \in \mathcal{U}\}$ is open in the Köthe space $E_{X \multimap Y}$. Since the map $f \mapsto \mathsf{M}(\psi \circ f \circ \varphi^{-1})$ is a linear isomorphism between $\mathcal{L}(E,F)$ and $E_{X \multimap Y}$, we turn in that way $\mathcal{L}(E,F)$ into a topological vector space, whose topology depends *a priori* on φ and ψ. So let $\sigma : E \to E_Z$ and $\tau : F \to E_T$ be some other linear homeomorphisms to Köthe spaces. Let $\theta = \varphi \circ \sigma^{-1}$, it is a linear homeomorphism from E_Z to E_X. One also sets $\eta = \tau \circ \psi^{-1}$, it is a linear homeomorphism from E_Y to E_T.

By lemma 24, the map $\theta \multimap \eta$ is a linear homeomorphism from $E_{X \multimap Y}$ to $E_{Z \multimap T}$. For $f \in \mathcal{L}(E,F)$, we have, by lemma 24 again,

$$(\theta \multimap \eta)(\mathsf{M}(\psi \circ f \circ \varphi^{-1})) = \mathsf{M}(\tau \circ f \circ \sigma^{-1})$$

and therefore $(\theta \multimap \eta)(\mathcal{U}_{\varphi,\psi}) = \mathcal{U}_{\sigma,\tau}$, hence $\mathcal{U}_{\sigma,\tau}$ is open (as $\theta \multimap \eta$ is open as an homeomorphism), so $\mathcal{U}$ is $\sigma\tau$-open. Symmetrically, if $\mathcal{U}$ is $\sigma\tau$-open then $\mathcal{U}$ is $\varphi\psi$-open. So we have defined a canonical topology on $\mathcal{L}(E,F)$, which depends only on the topologies of E and F (since these topologies determine the class of charts φ and ψ for E and F and since the topology we have defined on $\mathcal{L}(E,F)$ does not depend on a particular choice of charts φ and ψ as we have shown). Moreover, by construction, $\mathcal{L}(E,F)$ is an intrinsic Köthe space, since the function

$$\begin{array}{rcl}\hat{\mathcal{L}}(\varphi,\psi) : \mathcal{L}(E,F) & \to & E_{X \multimap Y} \\ f & \mapsto & \mathsf{M}(\psi \circ f \circ \varphi^{-1})\end{array}$$

is a linear homeomorphism (for the canonical topology of $\mathcal{L}(E,F)$) as soon as $\varphi : E \to E_X$ and $\psi : F \to E_Y$ are linear homeomorphisms: this holds by definition of the topology of $\mathcal{L}(E,F)$. Of course, $\mathbf{K}$ is an intrinsic Köthe space, and the topological vector spaces E' and $\mathcal{L}(E,\mathbf{K})$ are equal.

[7]The situation here is not as good as for the topological dual: we do not know yet any way of defining the topology of $\mathcal{L}(E,F)$ directly in terms of the topologies (and the bornologies) of E and F; in particular, the topology of uniform convergence on all equicontinuous sets is too coarse in general, see proposition 28.

We turn now the operation $(E,F) \mapsto \mathcal{L}(E,F)$ into a functor. Let E_1 and F_1 be intrinsic Köthe spaces, and let $f : E_1 \to E$ and $g : F \to F_1$ be linear continuous maps. If $h \in \mathcal{L}(E,F)$, then $g \circ h \circ f \in \mathcal{L}(E_1,F_1)$, and we set

$$\mathcal{L}(f,g)(h) = g \circ h \circ f\,.$$

Then the diagram

$$\begin{tikzcd} \mathcal{L}(E,F) \arrow[r, "\mathcal{L}(f,g)"] \arrow[d, "\hat{\mathcal{L}}(\varphi,\psi)"'] & \mathcal{L}(E_1,F_1) \arrow[d, "\hat{\mathcal{L}}(\varphi_1,\psi_1)"'] \\ E_{X\multimap Y} \arrow[r, "(\varphi\circ f\circ{\varphi_1}^{-1})\multimap(\psi_1\circ g\circ\psi^{-1})"] & E_{X_1\multimap Y_1} \end{tikzcd} \tag{17}$$

(where $\varphi_1 : E_1 \to E_{X_1}$ and $\psi_1 : F_1 \to E_{Y_1}$ are linear homeomorphisms) is commutative by lemma 24, and therefore, in particular, $\mathcal{L}(f,g)$ is continuous. In particular, when $F = F_1 = \mathbf{K}$ and $g = \mathrm{Id}$, one has of course $\mathcal{L}(f,\mathbf{K}) = {}^{\mathrm{t}}f$.

5.3 Tensor product

Due to the considerations above on topological duals of intrinsic Köthe spaces, $\mathcal{L}(E,F)$ comes equipped also with an intrinsic bornology which allows to define its topological dual as an intrinsic Köthe space. Therefore, we define in general

$$E \otimes F = \mathcal{L}(E,F')'$$

and this is an intrinsic Köthe space: if $\varphi : E \to E_X$ and $\psi : F \to E_Y$ are linear homeomorphisms then $\varphi \mathbin{\hat{\otimes}} \psi = \hat{\mathcal{L}}(\varphi,\psi')'$ is a linear homeomorphism from $E\otimes F$ to $E_{X\otimes Y}$.

Let E_1 and F_1 be intrinsic Köthe spaces, and let $f : E \to E_1$ and $g : F \to F_1$ be linear continuous maps. We define a map $f\otimes g : E\otimes F \to E_1 \otimes F_1$. So let $t \in E\otimes F = \mathcal{L}(E,F')'$ and let $h \in \mathcal{L}(E_1,F_1')$, we set

$$\langle (f\otimes g)(t), h\rangle = \langle t, {}^{\mathrm{t}}g \circ h \circ f\rangle\,.$$

In other terms, we have set $f\otimes g = {}^{\mathrm{t}}\mathcal{L}(f,{}^{\mathrm{t}}g)$, and so $f\otimes g$ takes its values in $E_1\otimes F_1$ and is a linear continuous map. Let $\varphi_1 : E_1 \to E_{X_1}$ and $\psi_1 : F_1 \to E_{Y_1}$ be linear homeomorphisms to some Köthe spaces. Then the following diagram commutes; this is a consequence of the commutation of diagram 17.

$$\begin{tikzcd} E\otimes F \arrow[r, "f\otimes g"] \arrow[d, "\varphi\mathbin{\hat{\otimes}}\psi"'] & E_1\otimes F_1 \arrow[d, "\varphi_1\mathbin{\hat{\otimes}}\psi_1"] \\ E_{X\otimes Y} \arrow[r, "(\varphi_1\circ f\circ\varphi^{-1})\otimes(\psi_1\circ g\circ\psi^{-1})"] & E_{X_1\otimes Y_1} \end{tikzcd}$$

This diagram, together with the corresponding properties of the tensor product of Köthe spaces, can be used for proving the various categorical properties of the tensor product (associativity, symmetry, Mac Lane pentagon). Given $u \in E$ and $v \in F$, the element $u\otimes v$ of $E\otimes F = \mathcal{L}(E,F')'$ is defined by $\langle u\otimes v, h\rangle = \langle h(u), v\rangle$, and we have of course $(f\otimes g)(u\otimes v) = f(u)\otimes g(v)$.

One can also prove, using the diagram above, that the category of intrinsic Köthe spaces and linear continuous maps is a $\star$-autonomous monoidal closed category.

5.4 Exponentials

Let E be an intrinsic Köthe space, and let $h : E \to \mathbf{K}$ be a function. If $\varphi : E \to E_X$ and $\psi : E \to E_Y$ are linear homeomorphisms, and if $h \circ \varphi^{-1}$ is entire (that is, belongs to $\mathcal{A}(X)$), then $h \circ \psi^{-1} = h \circ \varphi^{-1} \circ (\varphi \circ \psi^{-1})$ is also entire by proposition 33, since $\varphi \circ \psi^{-1} : E_Y \to E_X$ is linear and continuous.

So it is reasonable to say that a function $h : E \to \mathbf{K}$ is entire if, for some (and therefore for each) linear homeomorphism $\varphi : E \to E_X$ to a Köthe space, the function $h \circ \varphi^{-1}$ belongs to $\mathcal{A}(X)$. We denote by $\mathcal{A}(E)$ the vector space of all entire functions from E to $\mathbf{K}$. For any linear homeomorphism $\varphi : E \to E_X$ to a Köthe space, we have a linear isomorphism $\hat{\mathcal{A}}(\varphi) : \mathcal{A}(E) \to E_{(!X)^\perp}$ given by

$$\hat{\mathcal{A}}(\varphi)(h) = \Theta_X(h \circ \varphi^{-1})\,.$$

Through this isomorphism, $\mathcal{A}(E)$ inherits from $E_{(!X)^\perp}$ a linear topology for which $\hat{\mathcal{A}}(\varphi)$ becomes a linear homeomorphism (the inverse image by $\hat{\mathcal{A}}(\varphi)$ of the topology of $E_{(!X)^\perp}$). This topology does not depend on the particular linear homeomorphism φ we have used. Indeed, let $\psi : E \to E_Y$ be another linear homeomorphism and let $\theta = \varphi \circ \psi^{-1} : E_Y \to E_X$; this is a linear homeomorphism, and by proposition 33, the following diagram is commutative

$$\begin{tikzcd}
 & E_{(!X)^\perp} \arrow[dd, "(!\theta)^\perp"] \\
\mathcal{A}(E) \arrow[ur, "\hat{\mathcal{A}}(\varphi)"] \arrow[dr, "\hat{\mathcal{A}}(\psi)"'] & \\
 & E_{(!Y)^\perp}
\end{tikzcd}$$

where $(!\theta)^\perp$ is a linear homeomorphism. So $\mathcal{A}(E)$ has a canonical topology, and, endowed with that topology, $\mathcal{A}(E)$ is an intrinsic Köthe space.

Let E and F be intrinsic Köthe spaces and let $f : F \to E$ be linear and continuous. If $h \in \mathcal{A}(E)$, then $h \circ f \in \mathcal{A}(F)$, by proposition 33 again. The map $\mathcal{A}(f) : \mathcal{A}(E) \to \mathcal{A}(F)$ defined in that way makes the following diagram commutative

$$\begin{tikzcd}
\mathcal{A}(E) \arrow[r, "\mathcal{A}(f)"] \arrow[d, "\hat{\mathcal{A}}(\varphi)"'] & \mathcal{A}(F) \arrow[d, "\hat{\mathcal{A}}(\psi)"'] \\
E_{(!X)^\perp} \arrow[r, "(!(\varphi \circ f \circ \psi^{-1}))^\perp"] & E_{(!Y)^\perp}
\end{tikzcd}$$

and so is a linear continuous map. Therefore $E \mapsto \mathcal{A}(E)$ is a contravariant endofunctor on the category of intrinsic Köthe spaces and linear continuous maps. The dual functor $E \mapsto \mathcal{A}(E)'$ (and $f \mapsto {}^{\mathrm{t}}\mathcal{A}(f)$) will also be denoted by $E \mapsto\, !E$ and $f \mapsto\, !f$.

The various structural maps of the exponential can also be defined in an intrinsic way. As an example, we give an intrinsic version of the digging morphism $\mathrm{p}^E : !E \to\, !!E$. Given $\varphi : E \to E_X$ a linear homeomorphism, we *define* p^E as the dual of the unique map q making the following diagram commutative:

$$\begin{tikzcd} \mathcal{A}(\mathcal{A}(E)') \arrow[r, "q"] \arrow[d, "\hat{\mathcal{A}}(\hat{\mathcal{A}}(\varphi)')"'] & \mathcal{A}(E) \arrow[d, "\hat{\mathcal{A}}(\varphi)"] \\ E_{(!!X)^\perp} \arrow[r, "(\mathrm{p}^X)^\perp"] & E_{(!X)^\perp} \end{tikzcd}$$

This map q, which, by definition, is linear and continuous (since $\mathcal{A}(\hat{\mathcal{A}}(\varphi)')$ and $\hat{\mathcal{A}}(\varphi)$ are linear homeomorphisms), does not depend on φ: this results from the functoriality of the operation $X \mapsto {!X}$ and from the naturality of $X \mapsto \mathrm{p}^X$. Given $\Phi \in \mathcal{A}(\mathcal{A}(E)')$, one can check that the entire function $q(\Phi) : E \to \mathbf{K}$ is defined as follows. For $u \in E$, let $\delta_u \in \mathcal{A}(E)'$ be given by $\delta_u(h) = h(u)$ (the Dirac functional). That δ_u indeed belongs to $\mathcal{A}(E)'$ can be proved using a chart of E. Then, for $u \in E$, we have $q(\Phi)(u) = \Phi(\delta_u)$.

We consider now the central isomorphism $!(X \mathbin{\&} Y) = {!X} \otimes {!Y}$ and look for an intrinsic version of this isomorphism. Given E and F, two intrinsic Köthe spaces, this isomorphism is the dual of a function $\mathcal{A}(E \times F) \to \mathcal{L}(\mathcal{A}(E)', \mathcal{A}(F))$ which maps any entire map $h : E \times F \to \mathbf{K}$ to the linear and continuous map $\Phi_h : \mathcal{A}(E)' \to \mathcal{A}(F)$ given by $\Phi_h(S)(v) = \langle S, h_v \rangle$ where $h_v : E \to \mathbf{K}$ is the entire map defined by $h_v(u) = h(u, v)$. That h_v is entire is clear, therefore, for $S \in \mathcal{A}(E)'$, $\langle S, h_v \rangle \in \mathbf{K}$ is well defined. Moreover, the map $v \mapsto h_v$ is entire by cartesian closeness of $\mathcal{K}^!_{\mathbf{K}}$, and so the map $\Phi_h(S) : F \to \mathbf{K}$ is entire for all $S \in \mathcal{A}(E)'$. One can check that the diagram

$$\begin{tikzcd} \mathcal{A}(E)' \arrow[r, "\Phi_h"] \arrow[d, "\hat{\mathcal{A}}(\varphi)'"'] & \mathcal{A}(F) \arrow[d, "\hat{\mathcal{A}}(\psi)"] \\ E_{!X} \arrow[r, "k"] & E_{(!Y)^\perp} \end{tikzcd}$$

commutes, where $\varphi : E \to E_X$ and $\psi : F \to E_Y$ are charts for E and F. In this diagram, the map k is defined using h and the isomorphism between $!(X \mathbin{\&} Y)$ and ${!X} \otimes {!Y}$ (and of course the charts φ and ψ), and so Φ_h is linear and continuous. Last one checks that the map $h \mapsto \Phi_h$ is linear and continuous using another diagram involving charts. Conversely, given $\Phi \in \mathcal{L}(\mathcal{A}(E)', \mathcal{A}(F))$, one defines an entire map $h : E \times F \to \mathbf{K}$ by $h(u, v) = \Phi(\delta_u)(v)$ where $\delta_u \in \mathcal{A}(E)'$ is the Dirac functional at u. This establishes the announced intrinsic isomorphism between $\mathcal{A}(E \times F)$ and $\mathcal{L}(\mathcal{A}(E)', \mathcal{A}(F))$, this latter space being canonically isomorphic to $(\mathcal{A}(E)' \otimes \mathcal{A}(F)')'$. Now let $S \in {!E}$ and $T \in {!F}$, through this isomorphism, the action of $S \otimes T$ on an element h of $\mathcal{A}(E \times F)$ is given by

$$\langle S \otimes T, h \rangle = \langle \Phi_h(S), T \rangle \tag{18}$$

with the notations above. Of course, a similar formula holds, where the rôles of S and T are interchanged.

Remember that $!E$ is the topological dual of $\mathcal{A}(E)$, so it is similar to a space of distributions where the space of smooth functions with compact support or of rapidly decreasing smooth functions would have been replaced by a space of entire mappings (many very good texts are available on the theory of distributions, let us just mention Schwartz's famous book [Sch66]). With this intuition in mind, we can adopt a more convenient notation for the application of an element S of $\mathcal{A}(E)'$

to an element h of $\mathcal{A}(E)$: following the tradition, let us write $S_x(h(x))$ instead of $\langle S, h\rangle$. In this notation, x is a bound variable "of type E". Equation (18) can be written more conveniently

$$(S \otimes T)_{(x,y)}(h) = T_y(S_x(h(x,y)))$$

and of course we also have $(S \otimes T)_{(x,y)}(h) = S_x(T_y(h(x,y)))$.

From this "distribution" viewpoint, the map $\mathrm{m}_E : !E \otimes !E \to !E$ (the intrinsic version of the map m_X defined for Köthe spaces in section 3.4) can be seen as defining a convolution product. Considering m_E as a bilinear map $!E \times !E \to !E$, one can check that indeed

$$\langle \mathrm{m}_E(S,T), h\rangle = \langle S \otimes T, k\rangle \quad \text{where } k(u,v) = h(u+v)\,,$$

that is, with the notation above, $\mathrm{m}_E(S,T)_u(h(u)) = S_x(T_y(h(x+y)))$ (and also in the other way round) and this is exactly the definition of a convolution product of distributions. The neutral element for this operation is the inverse image of $e_0 \in E_{!X}$ through the chart $\hat{\mathcal{A}}(\varphi)' : !E \to E_{!X}$ (where φ is a chart of E), which is easily seen to be the Dirac functional at 0, $\delta_0 : h \mapsto h(0)$, as usual.

Example: Take $\mathbf{K} = \mathbf{R}$ and $E = \mathbf{K}$ (the implicit chart of E is the obvious map $E \to E_1$ where 1 is defined by $|1| = \{*\}$). Given a continuous map $f : \mathbf{R} \to \mathbf{R}$ *with compact support*, and $h \in \mathcal{A}(\mathbf{R})$, we can set

$$f^\vee(h) = \int_{\mathbf{R}} h(x) f(x) dx$$

defining a linear form $f^\vee : \mathcal{A}(\mathbf{R}) \to \mathbf{R}$. Let $M \in \mathbf{R}^+$ be an upper bound of $|f(x)|$ for $x \in \mathbf{R}$ and let $K \in \mathbf{R}^+$ be such that the support of f is included in $[-K, K]$. Then

$$\left|\int_{\mathbf{R}} x^n f(x) dx\right| \leq \frac{2MK}{n+1} K^n \leq CK^n$$

for a constant $C \in \mathbf{R}^+$, which depends only on f. So setting $A_n = \int_{\mathbf{R}} x^n f(x) dx$ for each $n \in \mathbf{N}$, we define a family of real numbers $(A_n)_{n \in \mathbf{N} = |!1|}$ which belongs to $E_{!1}$. The corresponding element of $\mathcal{A}(\mathbf{R})'$ is easily seen to be $f^\vee$. Moreover, it results from Weierstrass theorem (on each *compact* subset of $\mathbf{R}$, any continuous map can be uniformly approximated by a sequence of polynomials) that the map $f \mapsto f^\vee$ is injective.

So one can say that $!\mathbf{R}$ contains the space of continuous functions with compact support as a subspace[8]. Given now f and g two such functions, one can check that $\mathrm{m}_{\mathbf{R}}(f^\vee, g^\vee) = (f * g)^\vee$ where $f * g$ is the usual convolution product of functions:

$$(f * g)(x) = \int_{\mathbf{R}} f(x-y) g(y) dy\,,$$

which is a continuous function with compact support.

[8]This space $!\mathbf{R}$ contains actually all distributions with compact support: given a distribution T with compact support $K \subseteq \mathbf{R}$, we can define $T^\vee(h)$ for $h \in \mathcal{A}(R)$ by $T^\vee(h) = T(\varphi h)$ where φ is any smooth map with compact support which is constantly equal to 1 on K, and this mapping from the distributions with compact support into $!\mathbf{R}$ is injective by Weierstrass theorem again.

6 Some examples

The goal of this section is to give the interpretation of a few terms (proofs) as continuous linear or as entire mappings, and to compute a few derivatives of the interpretations of these terms.

The space corresponding to the type of booleans is $\mathbf{Bool} = 1 \oplus 1$, so that $E_{\mathbf{Bool}}$ is (isomorphic to) $\mathbf{K}^2$, its generic element will be denoted by $xe_{\mathbf{t}} + ye_{\mathbf{f}}$, taking $|\mathbf{Bool}| = \{\mathbf{t}, \mathbf{f}\}$. Consider the following term of (e.g.) boolean PCF, with one boolean parameter b:

$$\mathtt{if}\, b\, \mathtt{then}\, (\mathtt{if}\, b\, \mathtt{then}\, \mathbf{t}\, \mathtt{else}\, \mathbf{f})\, \mathtt{else}\, (\mathtt{if}\, b\, \mathtt{then}\, \mathbf{f}\, \mathtt{else}\, \mathbf{t})\,,$$

the semantics of this term will be an entire (indeed, polynomial) map from $E_{\mathbf{Bool}} = \mathbf{K}^2$ to itself. For computing this semantics, one considers first the linearized version of this term, with two boolean parameters b and c, which tests if its two arguments are equal:

$$\mathtt{if}\, b\, \mathtt{then}\, (\mathtt{if}\, c\, \mathtt{then}\, \mathbf{t}\, \mathtt{else}\, \mathbf{f})\, \mathtt{else}\, (\mathtt{if}\, c\, \mathtt{then}\, \mathbf{f}\, \mathtt{else}\, \mathbf{t})\,,$$

and which is interpreted by a linear map from $E_{\mathbf{Bool} \otimes \mathbf{Bool}}$ to $E_{\mathbf{Bool}}$ whose matrix A is given by

$$A_{(a_1,a_2),d} = \begin{cases} 1 & \text{if } a_1 = a_2 \text{ and } d = \mathbf{t}, \text{ or } a_1 \neq a_2 \text{ and } d = \mathbf{f} \\ 0 & \text{otherwise.} \end{cases}$$

Then we compose this map with the contraction, or more precisely with the linear map $C : !\mathbf{Bool} \to (\mathbf{Bool} \otimes \mathbf{Bool})$ obtained by composing the contraction map $!\mathbf{Bool} \to (!\mathbf{Bool} \otimes !\mathbf{Bool})$ with the tensorisation of the dereliction $!\mathbf{Bool} \to \mathbf{Bool}$ with itself. This matrix is given by $C_{\mu,(a_1,a_2)} = \delta_{[a_1,a_2],\mu}$. Therefore, the kernel of the linear map g defined by AC is spanned by the vectors e_μ with $\#\mu \neq 2$, and g is defined by

$$g(ue_{[\mathbf{t},\mathbf{t}]} + ve_{[\mathbf{f},\mathbf{f}]} + we_{[\mathbf{t},\mathbf{f}]}) = (u+v)e_{\mathbf{t}} + 2we_{\mathbf{f}}\,.$$

The entire map $f : E_{\mathbf{Bool}} \to E_{\mathbf{Bool}}$ which interprets our initial program is then given by

$$f(ue_{\mathbf{t}} + ve_{\mathbf{f}}) = g((ue_{\mathbf{t}} + ve_{\mathbf{f}})^!) = (u^2+v^2)e_{\mathbf{t}} + 2uve_{\mathbf{f}}\,.$$

This computation can also be performed more simply. First, for a Köthe space X, we define a conditional operator $\mathtt{if} : \mathbf{Bool} \multimap !X \multimap !X \multimap X$ by the following matrix:

$$\mathtt{if}_{b,\lambda,\rho,a} = \begin{cases} 1 & \text{if } b = \mathbf{t} \text{ and } \lambda = [a] \text{ and } \rho = [] \\ 1 & \text{if } b = \mathbf{f} \text{ and } \lambda = [] \text{ and } \rho = [a] \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathtt{if}$ can be seen as an entire map from $E_{\mathbf{Bool}} \times E_X \times E_X$ to E_X, which is linear in its first argument, and one checks easily that this map, still denoted by $\mathtt{if}$, is given by

$$\mathtt{if}(u\mathbf{t} + v\mathbf{f}, x, y) = ux + vy\,;$$

observe that this function is actually affine in its second and third argument. Then the function f can be computed as follows:

$$\begin{aligned} f(u\mathbf{t} + v\mathbf{f}) &= \mathtt{if}(u\mathbf{t} + v\mathbf{f}, \mathtt{if}(u\mathbf{t} + v\mathbf{f}, \mathbf{t}, \mathbf{f}), \mathtt{if}(u\mathbf{t} + v\mathbf{f}, \mathbf{f}, \mathbf{t})) \\ &= \mathtt{if}(u\mathbf{t} + v\mathbf{f}, u\mathbf{t} + v\mathbf{f}, u\mathbf{f} + v\mathbf{t}) \\ &= u(u\mathbf{t} + v\mathbf{f}) + v(u\mathbf{f} + v\mathbf{t}) \\ &= (u^2 + v^2)\mathbf{t} + 2uv\mathbf{f} \end{aligned}$$

It is noticeable that the interpretation of a perfectly standard syntactical object like the program we have just considered (which actually can also be considered as a proof in linear logic) is a matrix which has a coefficient different from 0 and 1. This is an effect of non-uniformity: observe that, in this example, the coefficient 2 corresponds to a component of f which would not exist in a uniform setting. It must be pointed out that this effect was already present in Girard's work on quantitative semantics [Gir88], since this model had no uniformity constraints, and Ryu Hasegawa showed in [Has97] how to compute effectively the entire coefficients associated to the interpretations of λ-terms using generating functions.

As a second example, we consider the map

$$\begin{array}{rcl} \mathbf{2} : E_{(!1)^\perp} & \to & \mathbf{K} \\ f & \mapsto & f(f(0)) \end{array}$$

which is a version of the "2 Church numeral" (0 is the zero of the base field). In this definition, we identify the space $E_{(!1)^\perp}$ with the space of entire functions defined on $\mathbf{K}$ with values in $\mathbf{K}$. This map $\mathbf{2}$ is entire by cartesian closeness of $\mathcal{K}^!_{\mathbf{K}}$, and we want to compute its derivative. Let $X = (!1)^\perp$, if $f \in E_X$ is given on $\mathbf{K}$ by $f(x) = \sum_n A_n x^n$, then $\mathbf{2}(f) = \sum_n A_n (A_0)^n$.

Let $T : E_{X\&X} \to \mathbf{K}$ be the "linearized" (this function actually is linear only in its first argument) version of $\mathbf{2}$, given by $T(f,g) = f(g(0))$, which is the composite of the evaluation map $\mathrm{ev} : E_{X\&1} \to E_1$ and $H : E_{X\&X} \to E_{X\&1}$, this latter map being linear and given by $H(f,g) = (f, g(0))$. The chain rule gives therefore, for $f, h, g, k \in E_X$:

$$\begin{array}{rcl} \dfrac{dT}{d(f,g)}(f,g)\cdot(h,k) & = & \dfrac{d\mathrm{ev}}{d(u,x)}(f,g(0))\cdot H(h,k) \\ & = & h(g(0)) + \dfrac{df}{dx}(g(0))\cdot k(0) \end{array}$$

by formulae (15) and (16), so

$$\frac{d\mathbf{2}}{df}(f)\cdot h = h(f(0)) + \frac{df}{dx}(f(0))\cdot h(0)$$

by formula (14). For instance, when $f(0) = 0$, we have $\frac{d\mathbf{2}}{df}(f)\cdot h = (1 + f'(0))h(0)$ (here, the linear application operation $A \cdot x$ is just scalar multiplication).

We leave to the reader the following computation. Let X be a Köthe space, and consider now the "true" Church numeral 2, namely the functional

$$\begin{array}{rcl} \mathbf{2} : E_{!X\multimap X} & \to & E_{!X\multimap X} \\ f & \mapsto & f \circ_{\mathcal{A}} f \end{array}$$

So $\mathbf{2}$ is entire by cartesian closeness of $\mathcal{K}^!_{\mathbf{K}}$, and $\frac{d\mathbf{2}}{df}$ is an entire map from $E_{!X\multimap X}$ to $E_{(!X\multimap X)\multimap(!X\multimap X)}$; show that this map is given by

$$\left(\frac{d\mathbf{2}}{df}(f)\cdot h\right)(x) = h(f(x)) + \frac{df}{dx}(f(x))\cdot h(x)\,.$$

For instance, $\frac{d\mathbf{2}}{df}(\mathrm{Id})$ maps any entire map $h : E_X \to E_X$ to $2h$.

An essential tool in programming is the *iteration* principle, which corresponds to the "`for`" loop of Pascal. As already explained, the type of natural numbers can be considered as a $\mathbf{N}$-indexed sum of the space 1, that is $|N| = \mathbf{N}$ and $E_N = \mathbf{K}^{(\mathbf{N})}$. So in the present setting, a "natural number" is a finite linear combination of standard natural numbers, with coefficients in $\mathbf{K}$ (we had the same phenomenon with booleans, which were combinations of $\mathbf{t}$ and $\mathbf{f}$). Given a Köthe space X, for each $k \in \mathbf{N}$, there is an entire map

$$\begin{array}{rcl} I_k : E_{(!X \multimap X)\&!X} & \to & E_X \\ (f, x) & \mapsto & f^k(x) \end{array}$$

where $f^k = f \circ_{\mathcal{A}} \cdots \circ_{\mathcal{A}} f$ (k times). That I_k is entire results from the cartesian closeness of $\mathcal{K}^!_{\mathbf{K}}$. Of course, $\widetilde{I_k}$ can be considered as a linear continuous map from $E_{(!X \multimap X)\otimes !X}$ to E_X. By definition of N as a direct sum, we can define a linear continuous map $\mathsf{It}_X : E_N \to E_{((!X \multimap X)\otimes !X) \multimap X}$ as follows:

$$\mathsf{It}_X(n, f, x) = \sum_{k \in \mathbf{N}} n_k f^k(x)\,,$$

this sum being finite, since an element n of E_N satisfies $n_k = 0$ for almost all values of $k \in \mathbf{N}$; this is the iteration functional in $\mathcal{K}_{\mathbf{K}}$. This iteration functional suggests to identify N with the algebra $\mathbf{K}[\xi]$ of polynomials with one indeterminate ξ and coefficients in $\mathbf{K}$. The successor function corresponds to "multiplication by the indeterminate ξ", and addition (defined with the above iteration functional) corresponds to polynomial multiplication. Note however that N is *not* an object of natural numbers in the usual categorical sense as it is not true in general (denoting by S the successor function) that $\mathsf{It}_X(\mathsf{S}(n), f, x) = f(\mathsf{It}_X(n, f, x))$; this equation holds only when n is a "standard" natural number, that is $n = e_k$ for some $k \in \mathbf{N}$. Nevertheless, the equation $\mathsf{It}_X(\mathsf{S}(n), f, x) = \mathsf{It}_X(n, f, f(x))$ always holds ("tail recursive" iteration).

Using a standard trick involving a cartesian product, this iteration functional can be used for defining a primitive recursion functional, so that Gödel system $\mathcal{T}$ can be interpreted in the category $\mathcal{K}^!_{\mathbf{K}}$, a powerful functional programming language enjoying strong normalization (see [GLT89]). Observe that *a priori* $\mathcal{K}^!_{\mathbf{K}}$ *is not* a model of PCF, as obviously entire functions do not always have fix-points: consider the map $x \mapsto x + 1$ from $\mathbf{K}$ to $\mathbf{K}$.

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