Math 670
Homework 4

Assigned 10/23, Due 11/2

1. Let $(\mathcal{M}, \perp)$ be a phase space with the commutative monoid $\mathcal{M} = (M, ., e)$. Prove that

$$(\,)^{\perp\perp} : \mathcal{P}(M) \longrightarrow \mathcal{P}(M)$$

   as defined in class is a *closure* operator.

2. Let $(\mathcal{M}, \perp)$ be a phase space as in Question 1. and $X, Y \subseteq M$. Show that:
   (a) $(X \otimes Y)^{\perp} = (X^{\perp} \parr Y^{\perp})$
   (b) $1 \multimap X = X$
   (c) $X \multimap Y = (X \otimes Y^{\perp})^{\perp}$
   (d) $X \oplus Y) = (X^{\perp} \& Y^{\perp})^{\perp}$
   (e) $!(X \& Y) = !X \otimes !Y$

3. Consider the completeness theorem which states that linear sequent calculus is sound and complete with respect to validity in phase spaces. Complete the soundness proof for MALL (no exponentials).

4. Complete the proof of $[\![A]\!] = Pr(A)$.

5. Let X, Y be coherent spaces. Prove the following identities and isomorphisms:
   (a) $(X \otimes Y)^{\perp} = (X^{\perp} \parr Y^{\perp})$
   (b) $1 \multimap X \cong X$
   (c) $(X \& Y)^{\perp} = X^{\perp} \oplus Y^{\perp}$
   (d) $X \otimes (Y \oplus Z) \cong (X \otimes Y) \oplus (X \otimes Z)$
   (e) $!(X \& Y) \cong !X \otimes !Y$