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Bisimulation Relations for Dynamical and Control Systems

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Abstract

In this paper we propose a new equivalence relation for dynamical and control systems called *bisimulation*. As the name implies this definition is inspired by the fundamental notion of bisimulation introduced by R. Milner for labeled transition systems. It is however, more subtle than its namesake in concurrency theory, mainly due to the fact that here, one deals with relations on manifolds. We further show that the bisimulation relations for dynamical and control systems defined in this paper are captured by the notion of abstract bisimulation of Joyal, Nielsen and Winskel (JNW). This result not only shows that our equivalence notion is on the right track, but also confirms that the abstract bisimulation of JNW is general enough to capture equivalence notions in the domain of continuous systems. We believe that the unification of the bisimulation relation for labeled transition systems and dynamical systems under the umbrella of abstract bisimulation, as achieved in this work, is a first step towards a unified approach to modeling of and reasoning about the dynamics of discrete and continuous structures in computer science and control theory.

1 Introduction

In the face of growing complexity of dynamical systems, various methods of complexity reduction are crucial to the analysis and design of such systems.

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Hence, the problem of equivalence of systems is of great importance to systems and control theory [20].

In the computer science community, and in particular in the field of concurrency theory, researchers have been working on various models and numerous equivalence notions for these models. Among these, process algebras and the notion of bisimulation are by now well established [13]. Category theory has been successfully used to understand and compare the multitude of models for concurrency by Winskel and Nielsen [22]. Related efforts include the categorically inspired framework for comparing models of computation in [12].

In [6], Joyal, Nielsen and Winskel proposed the notion of *span of open maps* in an attempt to understand the various equivalence notions for concurrency in an abstract categorical setting. They also showed that this abstract definition of bisimilarity captures the strong bisimulation relation of Milner [13]. Subsequently in [3] it was shown that abstract bisimilarity can also capture Hennessy's testing equivalences [4], Milner and Sangiorgi's barbed bisimulation [14] and Larsen and Skou's probabilistic bisimulation [11]. More recently, in [2], Blute et al. formulated a bisimulation relation for Markov processes on Polish spaces in this categorical framework, extending the work of Larsen and Skou. All this evidence further attests to the suitability of this abstract definition as an appropriate venue for formulation of bisimilarity concepts for dynamical, control, and hybrid systems. Other attempts to formulate the notion of bisimulation in categorical language, include the coalgebraic approach of [5,17].

In this paper we propose a new equivalence relation for dynamical and control systems (see also [15,19]) that we call bisimulation and further show that this equivalence relation is captured by the abstract bisimulation relation of JNW [6]. This extends the latter abstract framework to the continuous domains in control and systems theory. In this paper, our main focus, besides introducing a new equivalence relation for dynamical and control systems, is to establish a unification result for bisimulation of discrete and continuous systems. We postpone the discussion of the important issue of computational aspects of bisimulation for dynamical systems to subsequent work.

Our work also demonstrates the usefulness of a categorical language in transferring important and nontrivial notions between the fields of systems and control theory with a rich analytic [9], algebraic [8] and geometric structure [7], and automata-based models which are the main models in computer science. This is especially important for understanding the correct notions of equivalences for hybrid systems, a subject of our current research.

The rest of the paper is organized as follows: In Section 2, we briefly review the abstract formulation of the notion of bisimilarity due to JNW. Section 3, then provides the main application of this method in concurrency theory and recalls that the abstract bisimilarity captures Milner's strong bisimulation relation. The main theorems and results of our paper are contained in Sections 4 and 5 where we introduce and discuss bisimulation relations for dynamical and control systems respectively. We include in an Appendix the proofs of the results in Section 4, and leave the results of Section 5 without proof, as they follow the same line of reasoning as in Section 4.

2 Abstract Bisimulation

The notion of bisimilarity, as defined in [13], has turned out to be one of the most fundamental notions of operational equivalences in the field of process algebras. This has inspired a great amount of research on various notions of bisimulation for a variety of concurrency models. In order to unify most of these notions, Joyal, Nielson and Winskel gave in [6] an abstract formulation of bisimulation in a category theoretical setting.

The approach of [6] introduces a category of models where the objects are the systems in question, and the morphisms are simulations. More precisely, it consists of the following components:

- **Model Category:** The category $\mathbf{M}$ of *models* with objects the systems being studied, and morphisms $f : X \to Y$ in $\mathbf{M}$, that should be thought of as a simulation of system X in system Y.
- **Path Category:** The category $\mathbf{P}$, called *the path category*, where $\mathbf{P}$ is a subcategory of $\mathbf{M}$ of path objects, with morphisms expressing how they can be extended.

The path category will serve as an abstract notion of time. Since the path category $\mathbf{P}$ is a subcategory of $\mathbf{M}$ of models, time is thus modeled as a (possibly trivial) system within the same category $\mathbf{M}$ of models. This allows the unification of notions of time across discrete and continuous domains.

Definition 2.1 A *path* or *trajectory* in an object X of $\mathbf{M}$ is a morphism $p : P \to X$ in $\mathbf{M}$ where P is an object in $\mathbf{P}$.

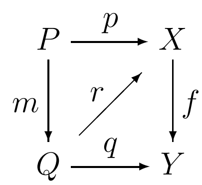
Let $f : X \to Y$ be a morphism in $\mathbf{M}$, and $p : P \to X$ a path in X, then clearly $f \circ p : P \to Y$ is a path in Y. Note that a path is a morphism in $\mathbf{M}$ and so is the map f and hence $f \circ p$ is a map in $\mathbf{M}$. This is the sense in which Y *simulates* X; any path (trajectory) p in X is matched by the path $f \circ p$ in Y.

The abstract notion of bisimulation in [6] demands a slightly stronger version of simulation as follows: Let $m : P \to Q$ be a morphism in $\mathbf{P}$ and let the diagram

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle f} \\ Q & \xrightarrow{q} & Y \end{array}$$

commute in $\mathbf{M}$, i.e., the path $f \circ p$ in Y can be extended via m to a path q in

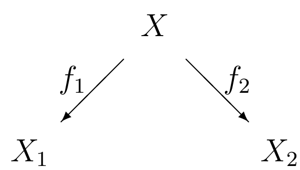
Y. Then we require that there exist $r : Q \to X$ such that in the diagram



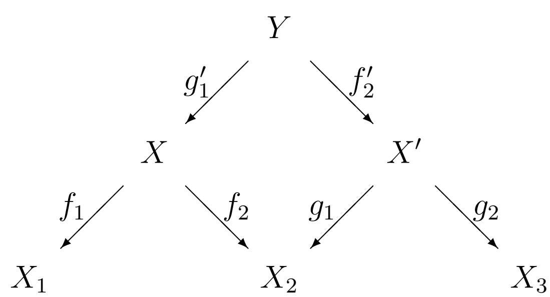
both triangles commute. Note that this means that the path p can be extended via m to a path r in X which matches q. In this case, we say that $f : X \to Y$ is $\mathbf{P}$-*open*. It can be shown that $\mathbf{P}$-open maps form a subcategory of $\mathbf{M}$.

Proposition 2.2 *Let* $\mathbf{M}$ *be a category and* $\mathbf{P}$ *be the subcategory of path objects. Then,* $\mathbf{P}$*-open maps in* $\mathbf{M}$ *form a subcategory of* $\mathbf{M}$*.*

The definition of $\mathbf{P}$-open maps leads to the notion of $\mathbf{P}$-bisimilarity. We say that objects X_1 and X_2 of $\mathbf{M}$ are $\mathbf{P}$-*bisimilar*, denoted $X_1 \sim_{\mathbf{P}} X_2$ iff there is a span of $\mathbf{P}$-open maps as shown below:



The relation of $\mathbf{P}$-bisimilarity between objects is clearly reflexive (identities are $\mathbf{P}$-open) and symmetric. It is also transitive *provided the model category* $\mathbf{M}$ *has pullbacks*, due to the fact that pullbacks of $\mathbf{P}$-open morphisms are $\mathbf{P}$-open (see [6] for a proof). Indeed suppose $X_1 \sim_{\mathbf{P}} X_2$ and $X_2 \sim_{\mathbf{P}} X_3$, then $X_1 \sim_{\mathbf{P}} X_3$ as can be seen from the following diagram.



Note that given X_1 and X_2 in $\mathbf{M}$, if there exists a $\mathbf{P}$-open morphism $f : X_1 \to X_2$, or a $\mathbf{P}$-open morphism $g : X_2 \to X_1$, then X_1 and X_2 are $\mathbf{P}$-bisimilar. The spans are (X_1, id_{X_1}, f) and (X_2, g, id_{X_2}) respectively.

We will see in the upcoming sections below that not all model categories that we consider have pullbacks of all morphisms, in particular the category of smooth manifolds and smooth mappings does not have pullbacks of *all* morphisms. We discuss the solution to this problem in the sections below.

3 Bisimulations of Transition Systems

We briefly illustrate how the framework described in Section 2 results in the usual notion of bisimulation in the sense of Milner [13], for details see [6]. The definitions of transition systems are slightly adapted from [22].

Definition 3.1 A *transition system* $T = (S, i, L, \longrightarrow)$ consists of the following:

- A set S of states with a distinguished state $i \in S$ called the *initial state*.
- A set L of labels
- A ternary relation $\longrightarrow \subseteq S \times L \times S$

We form the model category of transition systems $\mathbf{T}$ with objects being transition systems and a morphism $f : T_0 \to T_1$ with $T_0 = (S_0, i_0, L_0, \longrightarrow_0)$ and $T_1 = (S_1, i_1, L_1, \longrightarrow_1)$ given by $f = (\sigma, \lambda)$ where $\sigma : S_0 \to S_1$ with $\sigma(i_0) = i_1$ and $\lambda : L_0 \to L_1$ a partial function such that

(i) $(s, a, s') \in \longrightarrow_0$ and $\lambda(a)$ defined, implies $(\sigma(s), \lambda(a), \sigma(s')) \in \longrightarrow_1$ and

(ii) $(s, a, s') \in \longrightarrow_0$ and $\lambda(a)$ undefined, implies $\sigma(s) = \sigma(s')$.

In order to discuss the usual bisimilarity of transition systems we need to restrict our model category to the subcategory $\mathbf{T}_L$ of transition systems with the same label set L and morphisms of the form $f = (\sigma, id_L)$ which preserve all the labels. The category $\mathbf{T}_L$ has both binary products and pullbacks.

We define the path category $\mathbf{Bran}_L$ as the full subcategory of $\mathbf{T}_L$ of all synchronization trees with a single finite branch (possibly empty). Now a path in a transition system T in $\mathbf{T}_L$ is a morphism $p : P \to T$ in $\mathbf{T}_L$, with P an object in $\mathbf{Bran}_L$. Clearly this simply means that we look at the traces of the transition system. The $\mathbf{Bran}_L$-open maps in $\mathbf{T}_L$ are characterized as follows:

Proposition 3.2 *The* $\mathbf{Bran}_L$*-open morphisms of* $\mathbf{T}_L$ *are morphisms* $(\sigma, id_L) : T \to T'$ *with* $T, T' \in \mathbf{T}_L$ *such that:*

If $\sigma(s) \xrightarrow{a} s'$ *in* T'*, then there exists* $u \in S$*,* $s \xrightarrow{a} u$ *in* T *and* $\sigma(u) = s'$*.*

We now recall the strong notion of bisimulation introduced in [13]. Let T_0 and T_1 be two transition systems in $\mathbf{T}_L$, as above.

Definition 3.3 A binary relation $\mathcal{R} \subseteq S_0 \times S_1$ is a *strong bisimulation* if $(s, t) \in \mathcal{R}$ implies, for all $\alpha \in L$:

(i) Whenever $s \xrightarrow{\alpha} s'$ then, there is $t', t \xrightarrow{\alpha} t'$ and $(s', t') \in \mathcal{R}$,

(ii) Whenever $t \xrightarrow{\alpha} t'$ then, there is $s', s \xrightarrow{\alpha} s'$ and $(s', t') \in \mathcal{R}$.

Transition systems T_0 and T_1 are called strongly bisimilar, written $T_0 \sim T_1$, if $(i_0, i_1) \in \mathcal{R}$ for some strong bisimulation relation $\mathcal{R}$. The following theorem, proven in [6], shows that the abstract notion of $\mathbf{Bran}_L$-bisimilarity coincides with the traditional strong notion of bisimulation.

Theorem 3.4 ([6]) *Two transition systems (hence synchronization trees) over* *the same labeling set L, are* $\mathbf{Bran}_L$*-bisimilar iff they are* strongly bisimilar *in the sense of Milner [13].*

In the next sections, we consider the notion of $\mathbf{P}$-bisimilarity in the categories of dynamical and control systems.

4 Dynamical Systems

A dynamical system or vector field on a manifold M is a smooth section of the tangent bundle on M, that is a smooth map $X : M \to TM$ such that $\pi_M X = id_M$ where $\pi_M : TM \to M$ is the canonical projection of the tangent bundle onto the manifold M.

We proceed to define the model category $\mathbf{Dyn}$ of dynamical systems. The objects in $\mathbf{Dyn}$ are dynamical systems $X : M \to TM$ where M is smooth manifold. A morphism in $\mathbf{Dyn}$ from object $X : M \to TM$ to object $Y : N \to TN$ is a smooth map $f : M \to N$ such that

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle Y} \\ TM & \xrightarrow{Tf} & TN \end{array}$$

commutes. Thus related systems are said to be *f-related* [10]. The identity morphisms and composition are induced by those in the category $\mathbf{Man}$ of smooth manifolds and smooth mappings.

We proceed to define the path category $\mathbf{P}$ as the full subcategory of $\mathbf{Dyn}$ with objects $P : I \to TI$, where $P(t) = (t, 1)$ and I is an open interval of $\mathbb{R}$ containing the origin. Note that I is a manifold since it is an open set and it is also parallelizable (trivializable), that is $TI \cong I \times \mathbb{R}$. Observe that P represents the differential equation $dx(t)/dt = 1$ modeling a clock running on the interval I at unit rate. Also, any other choice $P'(t) = (t, c)$ with $0 \neq c \in \mathbb{R}$ instead of $P(t) = (t, 1)$ is isomorphic to P via $f : P \to P'$ with $f(t) = tc$.

Definition 4.1 A *path* or *trajectory* in a dynamical system $X : M \to TM$ is a morphism $c : P \to X$ in $\mathbf{Dyn}$, where P is an object in $\mathbf{P}$. More explicitly, a path c is a map $c : I \to M$ such that the following diagram commutes.

$$\begin{array}{ccc} I & \xrightarrow{c} & M \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle X} \\ TI & \xrightarrow{Tc} & TM \end{array}$$

This means that a path in X is a smooth map $c : I \to M$ for some open interval I such that $c'(t) = X(c(t))$ for all $t \in I$. Thus, a path in X is just an integral curve in M. Observe that given a path c in X, and $f : X \to Y$, then

$f \circ c$ is a path in Y. This is the sense of Y *simulating or over-approximating* X.

The next issue to understand is the meaning of path extension. Suppose $P : I \to TI$ and $Q : J \to TJ$ are objects in $\mathbf{P}$ with I, J open intervals in $\mathbb{R}$ containing the origin, and $m : P \to Q$. Then, m is a smooth map from I to J, such that $m'(t) = 1$ or $m(t) = t - t_0$ for some $t_0 \in I$ and for all $t \in I$.

We now introduce the following notation: let $\phi_X(x_1, x_2, t)$ denote the predicate that system X evolves from state x_1 to state x_2 in time t. Hence, $\phi_X(x_1, x_2, t)$ is true iff there is an open interval I in $\mathbb{R}$ containing the origin and an integral curve $c : I \to M$ such that $c(0) = x_1$ and $c(t) = x_2$. With this predicate, the characterization of $\mathbf{P}$-open maps is given by the following proposition.

Proposition 4.2 *Given the dynamical systems X on M and Y on N, $f : X \to Y$ is $\mathbf{P}$-open if and only if*

> *For any state $x_1 \in M$ of X and $t \in \mathbb{R}$, if $\phi_Y(f(x_1), y_2, t)$, then there exists $x_2 \in M$ such that $\phi_X(x_1, x_2, t)$ where $y_2 = f(x_2)$.*

In the particular case where vector fields are *complete*, that is solutions exist for all time, the previous proposition takes the following form.

Proposition 4.3 *Let X and Y on manifolds M and N respectively be complete vector fields. Then any $f : X \to Y$ is $\mathbf{P}$-open.*

Recall that by the general definition in Section 2, two objects X_1 and X_2 in the model category are bisimilar if there is a span of $\mathbf{P}$-open maps, that is an object X with $\mathbf{P}$-open maps $f_1 : X \to X_1$ and $f_2 : X \to X_2$. The bisimulation relation has to be an equivalence relation and for that purpose one requires the existence of pullbacks in the underlying model category. As is well known in differential geometry [1,10], in **Man** arbitrary pullbacks do not exist. Structure needs to be imposed on the maps in order to guarantee that pullbacks exist.

Definition 4.4 Given smooth manifolds M and N, a smooth map $f : M \to N$ and $x \in M$, let $T_x f : T_x M \to T_{f(x)} N$ be the derivative of f. We say that:

(i) f is an *immersion* at x if and only if the map $T_x f$ is injective.

(ii) f is a *submersion* at x if and only if the map $T_x f$ is surjective.

Definition 4.5 Let M, N be smooth manifolds and $f : M \to N$ be a smooth mapping and P be a submanifold of N. The map f is *transversal* on P iff for each $x \in M$ such that $f(x)$ lies in P, the composite

$$T_x(M) \xrightarrow{T_x f} T_{f(x)}(N) \to T_{f(x)}(N)/T_{f(x)}(P)$$

is surjective.

In particular, if for every $x \in M$, $T_x f$ is surjective, that is, if f is a submersion on M, then the composite in the definition above will be surjective and hence every submersion $f : M \to N$ is transversal on every submanifold P of N.

Definition 4.6 Given smooth maps $f : M \to P$ and $g : N \to P$, we say that f and g are transversal if $f \times g : M \times N \to P \times P$ is transversal on the diagonal subset Δ_P of $P \times P$.

Proposition 4.7 ([1]) *Let M and N be smooth manifolds and $f : M \to N$ a smooth map, then graph(f) is a smooth submanifold of $M \times N$.*

Proposition 4.8 *The category* **Man** *has transversal pullbacks.*

Obviously transversality is a sufficient condition and hence there are other pullbacks in the category **Man**. In view of this proposition we have the following result.

Proposition 4.9 *Pullback of submersions exists in* **Man**. *Moreover, the pullback of any submersion is a submersion.*

After all these preliminary results in the category **Man** of manifolds, we can finally get to our desired goal in the category of dynamical systems.

Proposition 4.10 *The category* **Dyn** *has binary products and transversal pullbacks.*

In this case, as we have seen above, we can only guarantee the transversal pullbacks. Hence we modify the definition for **P**-bisimulation to ensure that it becomes an equivalence relation. That is we require that there be a span of **P**-*open submersions*.

Definition 4.11 We say that two dynamical systems X_1 and X_2 are **P**-bisimilar if there exists a span of **P**-*open submersions* $(Z, f_1 : Z \to X_1, f_2 : Z \to X_2)$.

Note that if there exists a **P**-open submersion $f : X_1 \to X_2$, or a **P**-open submersion $g : X_2 \to X_1$, then X_1 and X_2 are **P**-bisimilar. The spans are (X_1, id_{X_1}, f) and (X_2, g, id_{X_2}) respectively. The existence of transversal pullbacks in **Dyn** allows us to show the following result.

Proposition 4.12 *The relation of* **P***-bisimilarity is an equivalence relation on the class of all dynamical systems.*

We proceed with the definition of bisimulation for dynamical systems, for this we need a notion of a well-behaved relation. We will show that bisimulation and **P**-bisimulation coincide. The following definition which seems to be new, is inspired by a relevant definition for equivalence relations on manifolds [1,18].

Definition 4.13 Let M and N be smooth manifolds and $\mathcal{R}$ be a relation from M to N, that is to say $\mathcal{R} \subseteq M \times N$. We say that $\mathcal{R}$ is *regular* iff

- $\mathcal{R}$ is a smooth submanifold of $M \times N$,
- the projection maps $\pi_1 : \mathcal{R} \to M$ and $\pi_2 : \mathcal{R} \to N$ are submersions.

Proposition 4.14 *Let X, Y and Z be smooth manifolds and $\mathcal{R} \subseteq X \times Y$ and $\mathcal{S} \subseteq Y \times Z$ be regular relations. Then $\mathcal{S} \circ \mathcal{R} \subseteq X \times Z$ is a regular relation.*

Definition 4.15 Given two dynamical systems X on M and Y on N, we say that a relation $\mathcal{R} \subseteq M \times N$ is a *bisimulation* relation iff

(i) $\mathcal{R}$ is a regular relation,

(ii) for all $(x, y) \in M \times N$ and $t \in \mathbb{R}$, $(x, y) \in \mathcal{R}$ implies
- if $\phi_X(x, x', t)$, there exists $y' \in N$ such that $\phi_Y(y, y', t)$ and $(x', y') \in \mathcal{R}$
- if $\phi_Y(y, y', t)$, there exists $x' \in M$ such that $\phi_X(x, x', t)$ and $(x', y') \in \mathcal{R}$

We say that two dynamical systems X and Y on manifolds M and N respectively are *bisimilar* if there exists a bisimulation relation $\mathcal{R} \subseteq M \times N$ such that for all $x \in M$ there exists a $y \in N$ with $(x, y) \in \mathcal{R}$ and vice-versa.

Theorem 4.16 *Given dynamical systems X and Y on manifolds M and N respectively, X and Y are bisimilar iff they are* **P***-bisimilar, i.e. $X \sim_{\mathbf{P}} Y$.*

The above theorem shows that the abstract notion of **P**-bisimilarity coincides with the expected and natural notion of bisimulation for dynamical systems. We now turn our attention to control systems.

5 Bisimulations of Control Systems

We define the model category **Con** as follows. Objects of **Con** are control systems over manifolds, a control system X over a manifold M is given by a pair (U_M, X_M) where $X_M : M \times U_M \to TM$ is a smooth map such that $\pi_M X_M = \pi_1$ with π_M the canonical tangent bundle projection. Here U_M is a smooth manifold called the *input space*. A morphism in **Con** from a control system $X = (U_M, X_M)$ to $Y = (U_N, Y_N)$ is given by a pair (ϕ_1, ϕ_2) of smooth maps with $\phi_1 : M \times U_M \to N \times U_N$ and $\phi_2 : M \to N$, such that

$$\begin{array}{ccc} M \times U_M & \xrightarrow{\phi_1} & N \times U_N \\ {\scriptstyle X_M}\downarrow & & \downarrow{\scriptstyle Y_N} \\ TM & \xrightarrow{T\phi_2} & TN \end{array} \qquad \begin{array}{ccc} M \times U_M & \xrightarrow{\phi_1} & N \times U_N \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ M & \xrightarrow{\phi_2} & N \end{array}$$

both commute. Thus related control systems are said to be (ϕ_1, ϕ_2)-related [16]. Note that since π_1 is a surjective map, ϕ_2 is uniquely determined given ϕ_1. The identity morphism $id_X : X \to X$ for an object X in **Con** is given by $id_X = (id_{M \times U_M}, id_M)$. Given $f : X \to Y$ and $g : Y \to Z$, the composite $gf : X \to Z$ is given by $gf = (g_1 f_1, g_2 f_2)$.

The path category **P** is defined as the full subcategory of **Con** with objects control systems (U_I, P_I) where U_I is the singleton space with trivial topology and thus $I \times U_I \cong I$. Hence $P_I : I \to TI$ with $P(t) = (t, 1)$ for all $t \in I$. I is an open interval of $\mathbb{R}$ containing the origin. Thus (I, P_I) is a well defined control system.

Definition 5.1 A *path* in a control system $X = (U_M, X_M)$ is then a morphism

$c = (c_1, c_2)$ in **Con** with $c_1 : I \to M \times U_M$ and $c_2 : I \to M$ such that

$$\begin{array}{ccc} I & \xrightarrow{c_1} & M \times U_M \\ {\scriptstyle P_I}\downarrow & & \downarrow{\scriptstyle X_M} \\ TI & \xrightarrow{Tc_2} & TM \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{c_1} & M \times U_M \\ {\scriptstyle id_I}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ I & \xrightarrow{c_2} & M \end{array}$$

commute.

This means that a path in X is a pair of smooth maps $c_1 : I \to M \times U_M$ and $c_2 : I \to M$ for some open interval I such that $c_2'(t) = X(c_2(t), u(t))$ for all $t \in I$, where $u(t) = \pi_2 c_1(t)$. Let (I, P_I) and (J, Q_J) be two path objects in **P** and $m = (m_1, m_2) : P \to Q$ be a path extension. Then from the diagram on the right above we get that $m_1 = m_2 : I \to J$ and then the diagram on the left coincides with the condition we had for dynamical systems. Thus a path extension $m = (m_1, m_2)$ is of the form $m_1 = m_2 : I \to J$, $m_1(t) = t - t_0$ for $t_0 \in I$ and for all $t \in I$.

Proposition 5.2 *The category* **Con** *has binary products and transversal pullbacks.*

We introduce the following notation: let $\phi_X(x_1, x_2, t)$ denote the predicate that system $X = (U_M, X_M)$ evolves from state x_1 to state x_2 in time t, under some input in U_M. Hence, $\phi_X(x_1, x_2, t)$ is true iff there is an open interval I of $\mathbb{R}$ containing the origin, a morphism $c = (c_1, c_2) : (U_I, P_I) \to X$ such that $c_2(0) = x_1$ and $c_2(t) = x_2$. The input deriving the system is given by $\pi_2 c_1 : I \to U_M$. Similarly to the case of dynamical systems, we characterize the **P**-open maps as follows.

Proposition 5.3 *Given the control systems $X = (M, X_M)$ and $Y = (N, Y_N)$, $f : X \to Y$ is* **P***-open iff*

> *For any state $x_1 \in M$ of X and $t \in \mathbb{R}$, if $\phi_Y(f(x_1), y_2, t)$, then there exists $x_2 \in M$ such that $\phi_X(x_1, x_2, t)$ where $y_2 = f(x_2)$.*

Definition 5.4 We say that two control systems X_1 and X_2 are **P**-bisimilar if there exists a span of **P**-*open submersions* $(Z, f_1 : Z \to X_1, f_2 : Z \to X_2)$.

Proposition 5.5 *The relation of* **P***-bisimilarity is an equivalence relation on the class of all control systems.*

We define the bisimulation relation for control systems, similarly to the case of dynamical systems.

Definition 5.6 Given two control systems $X = (U_M, X_M)$ and $Y = (U_N, Y_N)$, we say that a relation $\mathcal{R} \subseteq M \times N$ is a *bisimulation* relation iff

(i) $\mathcal{R}$ is a regular relation,

(ii) for all $(x, y) \in M \times N$ and $t \in \mathbb{R}$, $(x, y) \in \mathcal{R}$ implies
- if $\phi_X(x, x', t)$, there exists $y' \in N$ such that $\phi_Y(y, y', t)$ and $(x', y') \in \mathcal{R}$,
- if $\phi_Y(y, y', t)$, there exists $x' \in M$ such that $\phi_X(x, x', t)$ and $(x', y') \in \mathcal{R}$.

We say that two control systems X and Y as above are *bisimilar* if there exists a bisimulation relation $\mathcal{R} \subseteq M \times N$ such that for all $x \in M$ there exists a $y \in N$ with $(x, y) \in \mathcal{R}$ and vice-versa.

Theorem 5.7 *Given control systems $X = (U_M, X_M)$ and $Y = (U_N, Y_N)$, X and Y are bisimilar if and only if they are $\mathbf{P}$-bisimilar, i.e. $X \sim_{\mathbf{P}} Y$.*

The above theorem, shows how the categorical notion of bisimulation described in Section 2, also captures the expected notion of bisimulation for control systems.

6 Conclusions and Future Work

In this paper we propose a new equivalence notion for dynamical and control systems that we call bisimulation, we also prove that this definition is captured in both cases (dynamical and control systems) by the abstract bisimulation of JNW. As a natural extension of the present work, currently we are studying the formulation of bisimulation relation for hybrid dynamical systems and algebraic characterisations for bisimilarity that can lead to efficient computational methods. The abstract bisimilarity is also well connected with logic and game characterisations of bisimulation and presheaf semantics in the case of concurrency models [21]. Currently ongoing work includes the study and development of similar connections for dynamical, control and hybrid systems. In this way, we hope to get a natural specification logic for the description of properties of such systems. We hope that the present work can provide a framework general enough, thanks to category theoretical tools, in which to study a unified approach to the dynamics of discrete and continuous systems.

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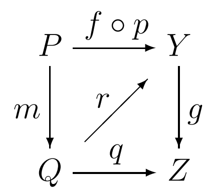
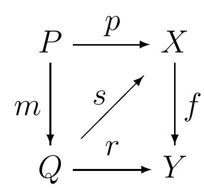
A Proofs

Proof. (**Proposition 2.2.**) Let X be an object in $\mathbf{M}$, we first show that $id_X : X \to X$ is a $\mathbf{P}$-open map. Let $p : P \to X$ and $q : Q \to X$ and $m : P \to Q$, where P and Q are path objects in $\mathbf{P}$. Assume also that $id_X p = qm$. Then let $r = q : Q \to X$: $id_X r = id_X q = q$ and $qm = p$. Now suppose, $f : X \to Y$ and $g : Y \to Z$ are $\mathbf{P}$-open maps, let $p : P \to X$ and $q : Q \to Z$, and $m : P \to Q$. Also assume that $(gf)p = qm$. As $g : Y \to Z$ is a $\mathbf{P}$-open map then there exists an $r : Q \to Y$ such that the triangles in the following diagram commute:

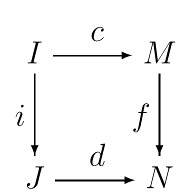


and as $f : X \to Y$ is $\mathbf{P}$-open, there exists a map $s : Q \to X$ making the triangles in the following diagram commute:



Now $(gf)s = g(fs) = gr = q$, using the second and the first diagrams for the last two equalities respectively. □

Proof. (**Proposition 4.2.**) Suppose $f : X \to Y$ is a $\mathbf{P}$-open map and $\phi_Y(f(x_1), y_2, t)$. Then there exists a path $d_1 : J_1 \to N$ such that $d_1(0) = f(x_1)$ and $d_1(t) = y_2$. Then, by the existence and uniqueness theorem for vector fields there exists a path $d : J \to N$ with J maximal such that $d(0) = f(x_1)$ and thus $J_1 \subseteq J$ and $d_1(t) = d(t)$ for all $t \in J_1$. Hence we have a path $d : J \to N$ such that $d(0) = f(x_1)$ and $d(t) = d_1(t) = y_2$. On the other hand, there is a path $c : I \to M$ with $c(0) = x_1$ for some open interval I of $\mathbb{R}$. Thus $fc(0) = f(x_1)$. By maximality, $I \subseteq J$ and $fc(t) = d(t)$ for all $t \in I$. Thus the following diagram (with i the inclusion map) commutes:



The $\mathbf{P}$-openness of f, then implies that there exists $r : J \to M$ such that $ri = c$ and $fr = d$. Hence we have $ri(0) = c(0) = x_1$ and $fr(t) = d(t) = y_2$. Let $x_2 = r(t)$, then clearly we have established $\phi_X(x_1, x_2, t)$.

Conversely, suppose that the condition of Proposition 4.2 holds, and given the path objects P and Q and $m : P \to Q$, with $p : P \to X$ and $q : Q \to Y$, $fp = qm$ holds. Note that as was observed earlier with $P : I \to TI$ and $Q : J \to TJ$, $m(t) = t - t_0$ for some $t_0 \in I$. Consider the point $p(t_0) \in M$, by the existence and uniqueness theorem for vector fields there exists an integral curve $\tilde{r} : \tilde{I} \to M$ with $\tilde{I}$ maximal with $\tilde{r}(0) = p(t_0)$. Suppose that there exists a $t \in J \setminus \tilde{I}$, then as q is a **Dyn**-morphism, we have $\phi_Y(q(0), q(t), t)$, but $\phi_Y(q(0), q(t), t) = \phi_Y(q(m(t_0)), q(t), t) = \phi_Y(f(p(t_0)), q(t), t)$ where the latter equality follows from assumption. Hence, there exists a point $x \in M$ such that $\phi_X(p(t_0), x, t)$ such that $f(x) = q(t)$. Hence there exists an integral curve $c : I_c \to M$ with $c(0) = p(t_0)$ and $c(t) = x$, and $t \in I_c \setminus \tilde{I}$ contradicting the maximality of $\tilde{I}$. Thus $J \subseteq \tilde{I}$. Now define r by $r = \tilde{r}|_J$. Clearly r is a **Dyn**-morphism. Now, $fr(0) = fp(t_0) = qm(t_0) = q(0)$ and hence $fr = q$. Also, $rm(t_0) = r(0) = p(t_0)$ and hence $rm = p$.

□

Proof. (Proposition 4.3.) Note that for complete vector fields any integral curve is defined on the whole of $\mathbb{R}$. Suppose $p : I \to M$ and $q : J \to N$ are paths and that $fp = qm$. Recall that $m : P \to Q$ is given by $m(t) = t - t_0$ for some $t_0 \in I$. Consider the point $p(t_0) \in M$, then by the existence and uniqueness theorem for vector fields there exists an integral curve $d : \mathbb{R} \to M$ such that $d(0) = p(t_0)$, define $r = d|_J : J \to M$. Clearly r is a **Dyn**-morphism. Now, $fr(0) = fp(t_0) = qm(t_0) = q(0)$ and hence $fr = q$. Similarly, $rm(t_0) = r(0) = p(t_0)$ and hence $rm = p$.

□

Proof. (**Proposition 4.8.**) Suppose M, N, P are smooth manifolds and $f_1 : M \to P$ and $f_2 : N \to P$ are smooth transversal maps. Form the fiber product of M and N on P, denoted $M \times_P N = \{(x, y) \in M \times N \mid f_1(x) = f_2(y)\}$. As f_1 and f_2 are transversal, $(f_1 \times f_2)^{-1}\Delta_P$ is a submanifold of $M \times N$, smoothness is induced by the differential structure of $M \times N$ [10]. The rest of the proof consists of checking the universal property of the pullback which follows from the set theoretical construction. □

Proof. (**Proposition 4.9.**) First note that the transversality condition given in the paper for a given $f_1 : M \to P$ and $f_2 : N \to P$ is equivalent to the following condition: for any $p \in P$ such that $p = f_1(x) = f_2(y)$ for some $x \in M$ and $y \in N$, $im(T_x f_1) + im(T_y f_2) = T_p P$ [10]. In other words, the tangent spaces on the left together must span the whole of $T_p P$. Now given that f_1 and f_2 are submersions we conclude that $im(T_x f_1) = im(T_y f_2) = T_p P$ for any $x \in M$ and $y \in N$ and hence transversality follows. To prove the second statement, recall that the pullback morphisms are projections restricted to $M \times_P N$, let $g_1 : M \times_P N \to N$ be the pullback of f_1 (see the diagram below), $Tg_1 : T(M \times_P N) \cong TM \times_{TP} TN \to TN$. Given any $(x, y) \in M \times_P N$, $T_{(x,y)} g_1 : T_x M \times_{T_{f_1(x)} P} T_y N \to T_y N$ is surjective as f_1 is a submersion. Hence g_1 is a submersion.

$$\begin{array}{ccc} M \times_P N & \xrightarrow{g_1} & N \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle f_2} \\ M & \xrightarrow{f_1} & P \end{array}$$

□

Proof. (**Proposition 4.10.**) Given the dynamical systems $X : M \to TM$ and $Y : N \to TN$, define $X \times Y : M \times N \to TM \times TN \cong T(M \times N)$ by $(X \times Y)(x, y) = (X(x), Y(y))$. The projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are morphisms in **Dyn** as can be easily seen from the definition.

Let X, Y and Z be dynamical systems on the manifolds M, N, P respectively and $f_1 : X \to Z$ and $f_2 : Y \to Z$. By assumption the maps $f_1 : M \to P$ and $f_2 : N \to P$ are transversal, so $M \times_P N$ is a smooth submanifold of $M \times N$. We define the dynamical system $W : M \times_P N \to T(M \times_P N) \cong TM \times_{TP} TN$, denoted $X \times_P Y$ by $W = X \times Y|_{M \times_P N}$. For this definition to be well-defined one has to ensure that for every point $(x, y) \in M \times_P N$, $(X \times Y)(x, y) \in TM \times_{TP} TN$, in other words one has to show that the vector field $X \times Y$ is tangent to the submanifold $M \times_P N$. We proceed by proving the equivalent statement: for any $(x, y) \in M \times_P N$ the flow of (x, y) along $X \times Y$ at any time t (for which the flow is defined), denoted $Fl_t^{X \times Y}(x, y)$ is in $M \times_P N$.

$$\begin{array}{ll} (Z \circ f_1)(x) = (Z \circ f_2)(y), & \text{as } (x, y) \in M \times_P N \\ T_x f_1 . X(x) = T_y f_2 . Y(y), & \text{as } f_1, f_2 \text{ are } \mathbf{Dyn}\text{-morphisms} \\ (\mathcal{L}_X f_1)|_x = (\mathcal{L}_Y f_2)|_y, & \mathcal{L}_X \text{ denotes the Lie derivative along the vector} \\ & \text{field } X \\ f_1(Fl_t^X(x)) = f_2(Fl_t^Y(y)), & \text{by integration} \\ Fl_t^{X \times Y}(x, y) \in M \times_P N, & \text{by definition.} \end{array}$$

The fact that $M \times_P N$ is a pullback in the category **Man** implies that W is a pullback in **Dyn**. □

Proof. (**Proposition 4.12.**) Reflexivity follows from the fact that id_X is a **P**-open submersion for any dynamical system X. Symmetry is trivial. For transitivity, suppose that $X_1 \sim_{\mathbf{P}} X_2$ and $X_2 \sim_{\mathbf{P}} X_3$. Then there exists the spans $(Z_1, f_1 : Z_1 \to X_1, f_2 : Z_1 \to X_2)$ and $(Z_2 : g_1 : Z_2 \to X_2, g_2 : Z_2 \to X_3)$. The pullback of f_2 and g_1 exist as these are submersions, denote these pullbacks by f_2' and g_1' respectively. We also know that f_2' and g_1' are **P**-open submersions. Moreover, composition of **P**-open maps is **P**-open and composition of submersions is a submersion. Thus we have the span of **P**-

open submersions $(Z, f_1 g'_1 : Z \to X_1, g_2 f'_2 : Z \to X_3)$ where Z is the vertex of the pullback square. $\Box$

Proof. **(Proposition 4.14.)** As $\mathcal{R}$ and $\mathcal{S}$ are regular relations the following pullback exists

$$\begin{array}{ccc} \mathcal{R} \times_Y \mathcal{S} & \xrightarrow{f_2} & \mathcal{S} \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle pr_1} \\ \mathcal{R} & \xrightarrow{pr_2} & Y \end{array}$$

Note that $\mathcal{R} \times_Y \mathcal{S} = \{(r,s) \,|\, pr_1(s) = pr_2(r)\} = \{(x,y,y',z) \,|\, y = y'\}$. Now consider $\mathcal{R} \times_Y \mathcal{S} \stackrel{pr_1 \times pr_2}{\longrightarrow} X \times Z$, then $\mathcal{S} \circ \mathcal{R} = (pr_1 \times pr_2)(\mathcal{R} \times_Y \mathcal{S})$. However, $pr_1 \times pr_2$ is a submersion and hence an open map. Thus $\mathcal{S} \circ \mathcal{R}$ is an open subset of $X \times Z$ and so a smooth submanifold of $X \times Z$. Furthermore, $\pi_1 : \mathcal{S} \circ \mathcal{R} \to X$ is given by $\mathcal{R} \times_Y \mathcal{S} \stackrel{f_1}{\longrightarrow} \mathcal{R} \stackrel{pr_1}{\longrightarrow} X$ which is a submersion. Similarly for $\pi_2 : \mathcal{S} \circ \mathcal{R} \to Z$. $\Box$

Proof. **(Theorem 4.16.)** Suppose that $X \sim_{\mathbf{P}} Y$ and $(Z, f : Z \to X, g : Z \to Y)$ is the span where $Z : P \to TP$. Note that $graph(f) \subseteq P \times M$ and $graph(g) \subseteq P \times N$ are regular relations. Consider the converse relation $\overline{graph(f)}$ and let $\mathcal{R} = graph(g) \circ \overline{graph(f)}$. By the proposition above $\mathcal{R}$ is regular. Let $(x,y) \in \mathcal{R}$ and $\phi_X(x,x',t)$, then there exists a $z \in P$ such that $(x,z) \in \overline{graph(f)}$ and $(z,y) \in graph(g)$, so $x = f(z)$. As f is a $\mathbf{P}$-open map, then there exist $z' \in P$ such that $\phi_Z(z,z',t)$ and $f(z') = x'$, i.e. $(z',x') \in graph(f)$. Let $y' = g(z')$, then $\phi_Y(g(z), g(z'), t) = \phi_Y(y,y',t)$ and $(x',y') \in \mathcal{R}$. Similarly, the other bisimilarity condition is satisfied.

Conversely, suppose that X and Y are bisimilar and $\mathcal{R}$ is the bisimulation relation. As it is regular, it is a smooth manifold. Consider the dynamical system $Z : \mathcal{R} \to T\mathcal{R}$ defined by $Z = (X \times Y)|_{\mathcal{R}}$. Note that as in Proposition 4.10 for Z to be well defined, one has to show that $X \times Y$ is tangent to the submanifold $\mathcal{R}$. We prove: for any point $(x,y) \in \mathcal{R}$, $Fl_t^{X \times Y}(x,y) = (Fl_t^X(x), Fl_t^Y(y)) \in \mathcal{R}$. Let $Fl_t^X(x) = x'$, then $\phi_X(x,x',t)$ and as $\mathcal{R}$ is a bisimulation relation, there exists y' such that $\phi_Y(y,y',t)$ and $(x',y') \in \mathcal{R}$, where $y' = Fl_t^Y(y)$. Also $pr_1 : \mathcal{R} \to M$ is a submersion. We need to show that pr_1 is $\mathbf{P}$-open. Let $\phi_X(pr_1(x,y), x', t) = \phi_X(x,x',t)$, then there exists y' such that $\phi_Y(y,y',t)$ and $(x',y') \in \mathcal{R}$, so $\phi_Z((x,y),(x',y'),t)$ and $pr_1(x',y') = x'$, so pr_1 is $\mathbf{P}$-open. Similarly for pr_2 and hence $(Z, pr_1 : Z \to X, pr_2 : Z \to Y)$ is a span of $\mathbf{P}$-open submersions and hence $X \sim_{\mathbf{P}} Y$. $\Box$